Solitons in ferromagnets and the KdV Hierarchy

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The higher order terms in the perturbative expansion that describes KdV solitons propagation in ferromagnetic materials are considered. They satisfy inhomogeneous linearized KdV equations, explicitly written down. The parity and homogeneity properties of the expansion show that half of these equations admit a zero solution. Long time propagation is investigated, through the consideration of the unbounded or secular solutions and a multi-time expansion. It is governed by all equations of the KdV Hierarchy. Major result of the paper is that the multi-scale expansion can be achieved up to any order with all its terms bounded, what is a necessary condition for its convergence.

Key words: KdV Hierarchy, Higher order KdV, KdV solitons, ferromagnets

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1 Introduction

Investigations of the higher order equations in multi-scale expansions are important, first from the point of view of the pure soliton theory, but also for applications of this theory to various domains in Physics. Perturbation theories for solitons have been developed [1], and have found applications, e.g. in the frame of the physics of optical solitons in fibers, related to optical telecommunications [2]. The higher order equations are an alternative way to deal with the corrections computed by these theories. From the purely theoretical point of view, it has been shown that the solvability of the higher order equations can in some sense be related to the problem of the complete integrability of the starting model [3],[4]. Furthermore, a physical interpretation of the equations of the Korteweg-de Vries (KdV) Hierarchy has been found by Kraenkel, Manna and Pereira, in the frame of water waves theory [5],[6],[7].

In the present article, this kind of expansion is applied to the problem of electromagnetic waves propagation in a ferro- or ferrimagnetic medium with a negligible conductivity. Although it doesn’t take into account the properly ferrimagnetic properties of this materials, the model discussed here describes in a quite realistic way the propagation of electromagnetic waves in ferrites [8],[9]. A wave propagation mode that gives rise to solitons in such a material has been discovered by the present author [10]. These solitons are described by the Korteweg-de Vries (KdV) equation, which is known to describe the propagation of long waves in shallow water [11]. Recall that the KdV equation is the first one that has been solved by the inverse scattering transform (IST) method [12],[13]. In fact, the partial differential system yielded by the Maxwell and the Landau equations is not linear at all. The weakly non-linear approximation, that leads to the asymptotic equations of nonlinear Schrödinger or Korteweg-de Vries type, is a priori not justified here. This differs strongly from the case of nonlinear optics. But, in the same way as the scientists of the seventies built experimental situations where a linear theory can be used, the weakly nonlinear situation of a relatively small signal added to some constant field is considered here. A signal comparable in its ampli-
tude to this constant field is only rejected by technical grounds: there is no physical reason for not considering it. These considerations make evident the interest for studying the higher order terms in the weakly nonlinear approximation for this physical situation. This is still true even if the model is a rough approximation from another point of view.

The approximation described by the KdV equation is valid up to some given propagation time only. After this time, the higher order terms get the same order of magnitude as the main one, and therefore can no more be neglected. These terms are called secular. An approximation on larger time intervals is obtained by incorporating these corrections into the main term. This appears as a velocity renormalization in [14], and through the consideration of the higher order time variables in the multiple time formalism of Kraenkel, Manna and Pereira [6]. The latter formalism, that involves the KdV hierarchy, is used in the present paper. The main term is shown to obey all equations of the KdV hierarchy. It is but clear that it cannot give a precise approximation of the real pulse during an infinite time: this would mean that the starting system was completely integrable, what is certainly not the case, even if it has never been proved. As in [15] and [16], the deviation from integrability should appear through a modification of the soliton shape and amplitude, and through the arising of radiation, especially during wave interaction. In the perturbative formalism, these effects are taken into account by the higher order terms. Conversely, the integrability of the starting equation appears through their vanishing [3]. When considering only the main time scale, at which the KdV equation yields a correct approximation, to take the higher order terms into account gives a finite but arbitrary accuracy. If propagation on a longer time is considered, the way in which the higher order terms depend on the higher order time variables must been clarified. This has only been done in a recent paper by the present author [17]: they obey the linearized KdV Hierarchy, and this allows to prove that no unbounded term does appear in the expansion. The paper gives the first example of application of this general theory.

It is organized as follows: in section 2 is described the model and summarized the algebraic part of the multi-scale expansion. The equations of the perturbative scheme are written down at an arbitrary order. We introduce some algebraic approach that is rather unusual in a physical frame, and close to that used for abstract mathematical studies, but it remains completely explicit. Section 3 is a study of the homogeneity and parity properties of the expansion, that are of major importance for the general theory to apply. We define adequate algebraic tools, in order to study and justify these physical properties. In particular, it is found that the equations of odd order have a zero r.h.s., despite the corresponding terms in the expansion do not vanish completely. This is a essential assumption in the general theory [17], which remained to be proved at least in some particular situation. Note that in [3],[5],[6] these terms did not appear at all. In section 4, the problem of secularities is presented and solved. The approximation obtained at each order is clarified. It is stated that the precision obtained depends on both the numbers of terms considered in the expansion, and on the order of magnitude of the propagation time. Then the evolution of the higher order terms with regard to the higher order time variables is considered. Section 5 yields a conclusion. The mathematical proofs and tools, a large part of which have been built specially, together with some more technical results, are presented in appendices: appendix A describes the derivation of the evolution equations up to any order in the multi-scale expansion. In appendix B, the linear part of the r.h.s. of these equations are computed explicitly, and in appendix C is given the proof of some statement given in section 3.

2 An adequate perturbative formalism

2.1 The Maxwell-Landau equations

We consider a classical model describing wave propagation in an infinite and isotropic ferromagnetic dielectric, immersed in a constant and uniform magnetic field. It consists in the Maxwell equations, with the simplest constitutive relation, linear and
isotropic, for the electric part, and the torque equation, also called Landau equation, that describes the evolution of the magnetization density \( \vec{M} \) in the magnetic field \( \vec{H} \):

\[
\partial_t \vec{M} = -\delta \mu_0 \vec{M} \wedge \vec{H} \quad (1)
\]

(\( \delta \) is the gyromagnetic ratio, and, in all this paper, we use the notation \( \partial_t \) for the partial derivation operator \( \frac{\partial}{\partial t} \)). Eq. (1), despite it neglects the inhomogeneous exchange interaction, anisotropy and damping, yields a good approximation for the description of wave propagation and interactions in ferro- or ferrimagnetic media [18],[9]. The electric field \( \vec{E} \) can be eliminated from the Maxwell equations to yield the following wave equation:

\[
-\nabla (\nabla \cdot \vec{H}) + \Delta \vec{H} = \frac{1}{c^2} \nabla^2 (\vec{H} + \vec{M}) \quad (2)
\]

where \( \Delta \) is the Laplacian operator, and \( c = 1/\sqrt{\mu_0 \varepsilon} \) is the speed of light based on the dielectric constant \( \varepsilon \) of the medium. \( \varepsilon \) is scalar and real, and defined so that the electric induction \( \vec{D} \) and the electric field \( \vec{E} \) satisfy the relation \( \vec{D} = \varepsilon \vec{E} \). In fact, the use of the second-order equation (2) is not suited to the mathematical techniques that we need to write the evolution equations up to any order in the multiscale expansion: the first order form (i.e., the Maxwell equations themselves) is more convenient.

After rescaling \( \vec{E}, \vec{H}, \vec{M}, t \) into \( \frac{\delta}{\varepsilon} \vec{E}, \frac{\delta \mu_0}{\varepsilon} \vec{H}, \frac{\delta \mu_0}{\varepsilon^2} \vec{M}, ct \), we get the following fundamental system:

\[
\begin{align*}
\partial_t \vec{E} &= \nabla \wedge \vec{H} \\
\partial_t \vec{H} &= -\nabla \wedge \vec{E} + \vec{M} \wedge \vec{H} \\
\partial_t \vec{M} &= -\vec{M} \wedge \vec{H}
\end{align*} \quad (3)
\]

System (3) can be written in the following matrix form:

\[
\partial_t u = \vec{A} \cdot \vec{x} + B(u, u) \quad (4)
\]

where the function \( u \) of the variables \( \vec{x} \) and \( t \) is valued in \( \mathbb{R}^9 \),

\[
u = \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix} \quad (5)
\]

\( \vec{A} = (A_x, A_y, A_z) \), where \( A_x, A_y, A_z \), are the three \( 9 \times 9 \) antisymmetrical matrices given by:

\[
A_s = \begin{pmatrix} 0 & R_s & 0 \\ -R_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (s = x, y, z) \quad (6)
\]

with:

\[
R_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]

\[
R_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Thus, \( \vec{A} \) is defined in such a way that:

\[
\vec{A} \cdot \nabla = \begin{pmatrix} 0 & \nabla \wedge 0 \\ -\nabla \wedge 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (8)
\]

\( B \) is a bilinear symmetrical mapping from \( \mathbb{R}^9 \times \mathbb{R}^9 \) into \( \mathbb{R}^9 \), defined by:

\[
B\left( \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix}, \begin{pmatrix} \vec{E}' \\ \vec{H}' \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \vec{M} \wedge \vec{H}' + \vec{M}' \wedge \vec{H} \end{pmatrix}
\]

\( B \) is defined so that:

\[
\vec{M} = \vec{M}_0 + \varepsilon \vec{M}_1 + \varepsilon^2 \vec{M}_2 + \cdots \quad (10)
\]

\( \vec{H} = \vec{H}_0 + \varepsilon \vec{H}_1 + \varepsilon^2 \vec{H}_2 + \cdots \)

where \( \vec{H}_0 \) is the constant field created in the medium by the external applied field, and \( \vec{M}_0 \) the corresponding magnetization density. The other terms are functions of the slow variables:

\[
\xi = \varepsilon (x - V t) \quad (11)
\]
It was found that \( \tilde{M}_1 \) and \( \tilde{H}_1 \) must be zero, and that \( \tilde{M}_2 \) and \( \tilde{H}_2 \) are proportional to some real function \( \varphi \), solution of the KdV equation:

\[
\partial_t \varphi + q \varphi \partial_x \varphi + r \partial_x^3 \varphi = 0 \quad (12)
\]

Explicit expressions for the constants \( q \) and \( r \), and for the velocity \( V \) were also found. Note that, as in usual KdV-type expansions, there is no carrier wave, fast oscillations or Fourier expansion here. Confusion with envelope solitons, as those described by the nonlinear Schrödinger (NLS) equation, in optical fibers, ferrites, water waves, or elsewhere, must be avoided. Here, in the same way, we expand \( u \) in a power series of \( \varepsilon \):

\[
u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \quad (13)
\]

We will see below that, as is described in ref. [6],[14], secularities (linear growth of the solutions with time) appear in the higher order terms. When considering propagation times with an order of magnitude larger than \( 1/\varepsilon^3 \), these secularities must be removed, and this is achieved, according to the multiple time formalism of Kraenkel et al., and to the general theory of [17], by imposing to the terms of higher order some particular dependency with regard to higher order time variables (the equations of the KdV Hierarchy). Let us introduce now these variables \( \tau_2, \tau_3, \ldots \), which are defined by:

\[
\tau_j = \varepsilon^{2j+1} t \quad (j \geq 1)
\]

(Thus, \( \tau = \tau_1 \)). The slow three-dimensional space variable writes:

\[
\vec{\xi} = (\xi, \eta, \zeta) = \varepsilon (\vec{x} - \vec{V} t)
\]

where \( \vec{V} \) is a speed vector to be determined. Reporting this expansion in eq. (4), and collecting the terms at each power in \( \varepsilon \), we get the following equation for every nonnegative integer \( p \) :

\[
-\vec{V} \cdot \nabla_{\vec{\xi}} u_{p-1} + \sum_{j \geq 1} \partial_{\tau_j} u_{p-2j-1} = \vec{A} \cdot \nabla_{\vec{\xi}} \varepsilon^{p-1} + \sum_{k+j=p} B(u_k, u_j) \quad (16)
\]

(We make the convention that \( u_p \) is zero if \( p \) is negative.) We will see that equation (16) can be reduced to an explicit recurrence formula giving 8 of the 9 degrees of freedom of \( u_p \), and a solvability condition, that yields, at each order, an evolution equation for this 9th degree of freedom. The evolution equation obtained for a rank \( p \) in eq. (16) governs the evolution of the term of order \( \varepsilon^{p-3} \).

### 2.3 The evolution equation at an arbitrary order

Before we solve eq. (16), we define the boundary conditions as follows: The first term \( u_0 = \begin{pmatrix} \vec{E}_0 \\ \vec{H}_0 \\ \vec{M}_0 \end{pmatrix} \) corresponds to the steady state induced by the constant and uniform applied magnetic field. The magnetization density is assumed to have its saturation value \( m = ||\vec{m}|| \), with an arbitrary direction, that we choose in the \( xy \)-plane without loss of generality: \( \vec{m} = \begin{pmatrix} m_x \\ m_y \\ 0 \end{pmatrix} \). The corresponding magnetic field is parallel to the magnetization, given by \( \alpha \vec{m} \), where \( \alpha \) is some positive real number, fixed. The applied electric field is zero. Let us call \( \vec{m} \) the 9-components vector \( \vec{m} = \begin{pmatrix} 0 \\ \alpha \vec{m} \\ \vec{m} \end{pmatrix} \), and assume that \( u_0 \rightarrow -\infty \vec{m} \), and that all other \( u_j \) and all the derivatives tend to zero as \( \xi \rightarrow -\infty \). Furthermore, we will assume that all the \( u_j \) are bounded, and restrict us to one spatial dimension, that is:

\[
\vec{V} = \begin{pmatrix} V \\ 0 \\ 0 \end{pmatrix} \quad \nabla_{\vec{\xi}} = \begin{pmatrix} \partial_{\xi} \\ 0 \\ 0 \end{pmatrix}
\]

(17)

It can seem rather unphysical to neglect any transverse variation of the wave profile, while we intend to study the longitudinal evolution up to any order in the perturbation theory. But it must be noticed that these two approximations are in fact independent one from each other. It is often possible to choose experimental conditions very close to the one-dimensional case. This will justify that we neglect the transverse variations. On the other hand, as soon as the values taken by the order parameter \( \varepsilon \)
become appreciable, the higher orders in perturbation theory become non negligible, whatever is the global precision expected.

Eq. (16) is reduced to a linear $3 \times 3$ algebraic system:
\[ L\vec{H}_n = \vec{W}_{n-1} \]  
(18)
in which the operator $L$ is defined by eq. (A10), and the r.h.s. $\vec{W}_{n-1}$ expresses in terms of the previous orders $u_1, \ldots u_{n-1}$ as in eq. (A11). Eq. (18) is equivalent to eq. (16), if we take into account the formulas:
\[ \vec{E}_n = \frac{1}{V} \sum_{j \geq 1} \int_{-\infty}^{\xi} \partial_{\xi_j} \vec{E}_{n-2j} - \frac{1}{V} R_{\xi} \vec{H}_n \]  
(19)
\[ \vec{M}_n = \sum_{j \geq 1} \int_{-\infty}^{\xi} \partial_{\xi_j} \vec{m}_{1}(u_{n-2j}) - \Gamma \vec{H}_n \]  
(20)
that give the rest of $u_n$ in terms of $\vec{H}_n$ and of the previous orders. The details of the computation and the definition of the notations are given in appendix A. It is seen that system (18) has a rank 2; thus at each order, $\vec{H}_n$ (or $u_n$) expresses in an unique way in terms of the previous orders, apart from a scalar function $\varphi_n$ to be determined (eqs. (A18) ff.). The searched evolution equations are the solvability conditions for eq. (18), which read:
\[ \vec{m} \cdot \vec{W}_n = 0 \]
for each $n$. We compute first the term involving $\varphi_n$: to obtain a nontrivial solution, we must impose some condition, that determines the velocity $V$. Then $\varphi_n$ vanishes from the equation, and we look at the term involving $\varphi_{n-1}$. Taking $n = 2$, we see that $\varphi_1$ is necessarily zero. Owing to the vanishing of some coefficients, this implies that the function $\varphi_{n-1}$ completely vanishes from the equation $\vec{m} \cdot \vec{W}_n = 0$. Thus this equation yields an evolution equation for $\varphi_{n-2}$. For the main term $n = 4$, it is the Korteweg-de Vries (KdV) equation:
\[ \partial_{\tau_1} \varphi_2 + q \varphi_2 \partial_\xi \varphi_2 + r \partial^3_\xi \varphi_2 = 0 \]  
(21)
$q$ and $r$ are some scalar coefficients, explicitly given by eqs. (A43-A44). For the higher order terms, $n = l+2 \geq 5$, it is the same equation, but linearized around the main order solution $\varphi_2$, and inhomogeneous. Its r.h.s. member $\Xi_l$ expresses in terms of the solutions $\varphi_2, \varphi_3, \ldots$ up to $\varphi_{l-1}$ of the previous orders, their derivatives and primitives. This equation reads:
\[ \partial_{\tau_1} \varphi_l + q \partial_\xi (\varphi_2 \varphi_l) + r \partial^3_\xi \varphi_l = \Xi_l(\varphi_2, \varphi_3, \ldots \varphi_{l-1}) \]  
(22)

### 2.4 About the formal solution

We can now write a formal solution:
\[ u^{(p)}(\xi) = \sum_{l=2}^{p} \varepsilon^l u_l(\xi, \tau_l) \]  
(23)
where $u_l$ is given by the recurrence formula (A19), and the $\varphi_n$ functions are solutions of the equations (21) and (22). $u^{(p)}_\varepsilon$ is obviously expected to give some approximation of the exact solution. In which sense this must be precise. Convergence theorems have recently been given for the multi-scale expansions leading to the nonlinear Schrödinger equation, and to the Davey-Stewartson system, in a general frame including the model considered in the present paper [19]. Leaving by side many technical points, we notice that these theorems state: for $\varepsilon$ small enough, there exists some $T$, such that for times $t \leq T/\varepsilon$ (the quoted paper considers 3 time variables, $t/\varepsilon$, $\tau$, and $\xi t$), the difference between the main term of the expansion and the exact solution is uniformly bounded by a $O(\varepsilon)$. Thus the convergence is proved only when the slowest variable belongs to some bounded interval. We do not intend to give any convergence proof in the present paper, and even we will not make any attempt to specify the norms, but we feel useful to clarify what kind of convergence can be eventually conjectured here, in order to clarify the meaning of the expansion. Here it can be expected that the approximate solution $u^{(p)}_\varepsilon$ differs from the exact solution for some $O(\varepsilon^{p+1})$, on a time interval bounded by some $T/\varepsilon^3$.

Difficulties arise when solving eqs. (21) and (22), that a priori forbid to propagate the solution on longer times. They are discussed in section 4. This discussion uses several algebraical and symmetry properties of the expansion, that give the matter of the next section.
3 Homogeneity properties of the r.h.s. members $\Xi_n$

3.1 Homogeneity

When writing the equations of the perturbative scheme (16), we selected the coefficients of some given power of $\varepsilon$, thus these coefficients must have some homogeneity properties. These are essential in the general frame of the multi-time formalism. Indeed, the introduction of the KdV Hierarchy is based on the fact that it is the only system that both is compatible and possesses the same homogeneity properties as the terms of the perturbative expansion. Let us first define this homogeneity in a rigorous mathematical way. $\varphi_k$ appears in the expression of $u_k$, which is the coefficient of order $\varepsilon^k$ in the expansion of the fields. $\partial_t$ is defined as $\partial_x = \varepsilon \partial_k$, and, while $\partial_t = -\varepsilon V \partial_x + \varepsilon^3 \partial_{\tau_1} + \varepsilon^5 \partial_{\tau_2} + \ldots$, each $\partial_{\tau_i}$ is the coefficient of some factor of order $\varepsilon^{2j+1}$.

Every term $A$ in the expression of the r.h.s. member $\Xi_n (\varphi_1, \varphi_2, \ldots, \varphi_n)$ of the linearized KdV equation (22) is a polynomial expression in the functions $\varphi_1$, $\varphi_2$, ..., and the operators $\partial_t$, $\left(\int_{-\infty}^{\xi} \partial_{\tau_2}\right)$, $\left(\int_{-\infty}^{\xi} \partial_{\tau_2}\right)$, ... This is easily checked using the recurrence formulas (A19)(A11). The important feature is that each time derivation is coupled with a spatial primitivation, and reciprocally. We call $\beta_k = \beta_k (A)$ the total exponent of $\varphi_k$ in $A$ (for each $k$), $\theta = \theta (A)$ the one of $\partial_t$, and $\alpha_j = \alpha_j (A)$ the total exponent of $\left(\int_{-\infty}^{\xi} \partial_{\tau_2}\right)$ (for each $j$). Then $A$ would appear as the coefficient of:

$$\varepsilon^{\beta_1} \varepsilon^{\beta_2} \ldots \varepsilon^{\beta_k} \times \prod_{j \geq 1} \left(\frac{1}{\varepsilon^{2j+1}}\right)^{\alpha_j} = \varepsilon^{d(A)}$$

(24)

where the exponent $d(A)$ is equal to:

$$d(A) = \sum_{j \geq 1} 2j \alpha_j (A) + \theta (A) + \sum_{k \geq 1} k \beta_k (A)$$

(25)

$u_n$ is a term of order $\varepsilon^n$, thus is homogeneous in relation to the “$d$-degree” $d$, with:

$$d(u_n) = n$$

(26)

In the same way, the r.h.s. $\tilde{W}_n$ of system (18) appears there with index $n = p - 1$, it is thus homogeneous with $d$-degree:

$$d(\tilde{W}_n) = p = n + 1$$

(27)

$\Xi_n$ is computed from $\tilde{W}_{n+2}$, thus is homogeneous with $d$-degree:

$$d(\Xi_n) = n + 3$$

(28)

The use of an explicit mathematical definition for this homogeneity allows to prove formulas (26) to (28) in a more rigorous way. They are indeed proved by mathematical induction using recurrence formulas (A19) and (A11). This proof does not present any special difficulty and is left to the reader.

3.2 The terms of odd order

Currently, in KdV-type expansions, the odd-order terms are taken to be zero in the following sense: in the formula analogous to (10), only even order terms are written. As an example, in ref. [14], where the quantity that we call $\varepsilon$ is called $\varepsilon^{\frac{1}{2}}$, there are only integer exponents in the expansion of the “fields”. Here, we can see that the terms $u_n$ with odd values of integer $n$ are not zero. But the parity of the case studied by Kodama and Taniuti is yet partly conserved in the following sense: if no nonzero initial condition is imposed at the odd orders, the r.h.s. $\Xi_n$ of the higher order equations and the corresponding solutions $\varphi_n$ are zero at all odd orders. This feature is not trivial, and in the general theory it must be taken as an hypothesis. In the case of ferromagnet it is possible to prove it, using homogeneity properties of the expansion, in the following way.

Let us call $\Pi$ the $xy$ plane, and $\Delta$ the $z$ axis. The proof rests on the following property: most of the operators that intervene in the algebraic computation permute $\Pi$ and $\Delta$, and appear with a certain kind of parity. The “degree” associated with this parity, which we will call “$\delta$-degree” has a rather exotic definition, because it depends simultaneously on the degree relative to the $\varphi_j$ and on the belonging of the term to $\Pi$ or $\Delta$. This definition is imposed by the algebraic properties of the problem. First we

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define a $\Pi$-degree:

$$\delta_\Pi(\vec{v}) = \begin{cases} 0 & \text{if } \vec{v} \in \Pi \\ 1 & \text{if } \vec{v} \in \Delta \end{cases}$$

(29)

$(\delta_\Pi(\vec{v})$ is not defined in the other cases.) For the 9-components vectors, we take an analogous definition, but with the following particularity:

$$\delta_\Pi(\begin{pmatrix} \vec{e} \\ \vec{h} \\ \vec{m} \end{pmatrix}) = 0 \text{ if } \vec{h} \in \Pi \text{ and } \vec{m} \in \Pi, \text{ but } \vec{e} \in \Delta$$

(and in the same way, $\delta_\Pi(\begin{pmatrix} \vec{e} \\ \vec{h} \\ \vec{m} \end{pmatrix}) = 1 \text{ if } \vec{h} \in \Delta, \vec{m} \in \Delta, \text{ and } \vec{e} \in \Pi$) For a polynomial expression $A$, in $\varphi_1, \varphi_2, \varphi_3...$, belonging to $\mathbb{R}^3$ or to $\mathbb{R}^9$, the $\delta$-degree will be defined by:

$$\delta(A) = \sum_{k \geq 1} k \beta_k(A) + \delta_\Pi(A)$$

(30)

$(\beta_k$ is defined as above.) We show by mathematical induction in appendix C that:

$$\delta(u_k) \equiv k \ [2]$$

(31)

$i.e.$ that these two quantities have the same parity. Let us assume that the $\varphi_n$, with an odd value of $n$, are not all equal to zero. We call $n_0$ the smallest odd value of $n$ for which $\varphi_{n_0} \neq 0$. The equation that governs the evolution of $\varphi_{n_0}$ is $\vec{m} \cdot \vec{W}_{n_0+2} = 0$. Consider a monomial $\vec{A}$ of $\vec{W}_{n_0+2}$ which doesn’t contain $\varphi_{n_0}$. Because of (C60), and because $n_0$ is odd, $\delta(\vec{A}) = \delta(\vec{W}_{n_0+2})$ is odd. On the other hand, only $\varphi_k$ with even values of $k$ appear in $\vec{A}$ (the other are zero). Thus, owing to the definition of the $\delta$-degree, is found that:

$$\delta(\vec{A}) \equiv \delta_\Pi(\vec{A}) \equiv 1 \ [2]$$

Thus $\vec{A}$ belongs to $\Delta$ and $\vec{m} \cdot \vec{A} = 0$. This yields finally:

$$\Xi_{n_0} = 0$$

(32)

$\varphi_{n_0}$ is thus solution of an homogeneous linear equation; if the initial condition is zero, $\varphi_{n_0}$ is zero. Thus all the $\varphi_n$, and all the $\Xi_n$, with odd values of $n$ are zero.

### 3.3 Parity properties

Let us present some other interesting symmetry property of the perturbative expansion, related to the symmetry of the basic equations (1) and (3), that corresponds to a change of the exterior field in its direction, for the opposite one. Mathematically, the considered transformation consists in replacing the vector parameter $\vec{m}$ by $-\vec{m}$. It is clear that nothing is modified if the propagation direction changes in the same time. The transform $\vec{m} \mapsto -\vec{m}$ is thus equivalent to the transform $V \mapsto -V$. Beside this, it is easily seen that eqs. (1) and (3) are not modified by the following transform $\sigma$:

$$\vec{m} \mapsto \vec{m}^\sigma = -\vec{m}$$

$$u = \begin{pmatrix} \vec{E} \\ \vec{H} \\ \vec{M} \end{pmatrix} \mapsto u^\sigma = \begin{pmatrix} -\vec{E} \\ -\vec{H} \\ -\vec{M} \end{pmatrix}$$

(33)

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} x^\sigma \\ t^\sigma \end{pmatrix} = \begin{pmatrix} -x \\ -t \end{pmatrix}$$

(34)

We have, for each $j$:

$$u_j = u_j(\varepsilon(x-Vt), \varepsilon^3t, \varepsilon^5t, \ldots)$$

The transform $\sigma$ (33) conserves this relation only if $\varepsilon$ changes also into $\varepsilon^\sigma = -\varepsilon$. Then $\xi$ is unchanged and $\tau_j$ is transformed into $-\tau_j$, and $\varphi_k$ into $(-1)^k \varphi_k$. The action of the transform $\sigma$ on the perturbative expansion can thus be defined algebraically in the following way:

$$m_x \mapsto -m_x$$

$$m_t \mapsto -m_t$$

$$\partial_\xi \mapsto +\partial_\xi$$

(35)

for all $j \geq 1$:

$$\int_{-\infty}^{\xi} \partial_\tau_j \mapsto +\int_{-\infty}^{\xi} \partial_\tau_j$$

for all $k \geq 1$:

$$\varphi_k \mapsto (-1)^k \varphi_k$$

From recurrence formulas (A19) and (A11), we show by induction that:

$$u_k^\sigma = (-1)^{k+1}u_k$$

$$\vec{W}_n^\sigma = (-1)^{n+1}\vec{W}_n$$

(36)
Thus, for all $n \geq 1$:

$$q^\sigma = q \quad r^\sigma = r$$

and

$$\Xi_n^\sigma = (−1)^n \Xi_n$$

(37)

As seen above, $\Xi_n^1 = 0$ for all odd value of $n$, thus, from (37), we see that the transform $\varphi$ doesn't modify anything in the whole perturbative expansion.

4 Boundedness of the expansion terms

4.1 The problem of secularities

It is well known that the KdV equation (21) is completely integrable, i.e., that the Cauchy problem for it can be solved by use of the Inverse Scattering Transform (IST) method [12],[13],[20]. The IST method gives also explicit formulas for the resolution of the linearized KdV equation (22) [14],[21]. Despite this latter equation is linear, the use of the IST method gives also explicit formulas for the resolution of the linearized KdV equation (22) [14],[21].

4.2 The second order equation and the KdV Hierarchy

What happens at second order

Using the above homogeneity properties, it has been shown that, if the corresponding initial conditions are zero, all the $\Xi_n$ and all the $\varphi_n$ with odd values of the integer $n$ are zero. Then the second equation of the perturbative expansion is the equation (22) for $\varphi_4$. The approximate solution corresponding to this step of the perturbative expansion reads:

$$u_4^{(4)} = \varepsilon^4 \varphi_2 \bar{u}_1 + \varepsilon^3 u_3^{(0)}(\varphi_2)$$

$$+ \varepsilon^4 \left( \varphi_4 \bar{u}_1 + u_4^{(0)}(\varphi_2) \right)$$

(39)

We use the notations defined in appendix A, eq. (A19) and following. It can be conjectured that $u_4^{(4)} - u \in O(\varepsilon^5)$ for $t \leq T/\varepsilon^3$, for some $T$, and $\varepsilon$ small enough. $\varphi_4$ is secular, thus this approximation is not valid for larger $t$. The secularities must thus be removed.

According to [21], the secular solutions of the linearized KdV equation are removed when the linear
terms vanish from the r.h.s. members of these equations. This involves the computation of their linear part. The r.h.s. $\Xi_4(\varphi_2)$ of the second equation of the perturbative expansion is given by formula (B53); The indices $(\alpha_j)_{j \geq 1}$, $k$ in this sum take the following values only: $(\alpha_j)_{j \geq 1} = (0, \ldots, 5)$; $(1, 0, \ldots, 3)$; $(2, 0, \ldots, 1)$; $(0, 1, 0, \ldots, 1)$. Furthermore, $\Xi((0, 1, 0, \ldots, 1)) = -1$ (eq. (B55)), and, as $\varphi_2$ satisfies the KdV equation (21), we have:

$$\int_{-\infty}^{\xi} \partial_\tau \varphi_2 = -r \partial_\xi^2 \varphi_2 + O_2$$

Thus the expression of $\Xi_4(\varphi_2)$ reduces to:

$$\Xi_4(\varphi_2) = -\partial_\tau \varphi_2 - r_2 \partial_\xi^2 \varphi_2 + O_2 \quad (40)$$

where $O_2$ designates here an expression in $\varphi_2$ without linear terms, and the coefficient $r_2$ is given by:

$$-r_2 = \Xi((2, 0, \ldots, 1))(-r)^2 + \Xi((1, 0, \ldots, 3))(-r) + \Xi((0, \ldots, 5)) \quad (41)$$

The removal of the secular-producing terms is achieved, according to the multi-time formalism of Kraenkel, Manna, and Pereira [6], and to the general theory of [17], by imposing the following evolution equation for $\varphi_2$, relative to the second-order time variable $\tau_2$:

$$\frac{-1}{16r_2} \partial_\tau \varphi_2 = \partial_\xi \mathcal{L}^2 \varphi_2 \quad (42)$$

with:

$$\mathcal{L} = \frac{1}{4} \partial_\xi^2 - \frac{q}{6r} \varphi_2 + \frac{q}{12r} \int_{-\infty}^{\xi} d\xi (\partial_\xi \varphi_2). \quad (43)$$

It is easily checked that $\Xi_4(\varphi_2)$ doesn’t contain any linear term more.

In fact, for $t \leq T/\varepsilon^3$, this procedure does not modify $u_\varepsilon^{(4)}$ except for some $O(\varepsilon^4)$. On the other hand, $u_\varepsilon^{(2)} = \varepsilon^2 \varphi_2(\xi, \tau_1, \tau_2) u_1$ is expected to yield an approximation in the sense: $\left| u_\varepsilon^{(2)} - u \right|_{\mathcal{L}^\infty} = O(\varepsilon^3)$ for $t \leq T/\varepsilon^5$. Notice that the $\tau_2$ evolution of $\varphi_4$ is a priori not known, what implies that the approximate value $u_\varepsilon^{(4)}$ of $u$ to within a $O(\varepsilon^5)$, is no more valid when $t$ reaches the order of magnitude of $1/\varepsilon^5$.

**Generalization by means of the KdV Hierarchy**

The linear, thus the secular-producing, terms in $\varphi_2$ are removed by the same way at each order. We impose for each $p \geq 2$:

$$\frac{-1}{(-4)^p r_p} \partial_\tau \varphi_2 = \partial_\xi \mathcal{L}^p \varphi_2 \quad (44)$$

with $\mathcal{L}$ as above. The scaling coefficient $r_p$ is defined by $r_1 = r$ and the following recurrence formula:

$$r_{p+1} = \sum_{(\alpha_j)_{1 \leq j \leq p-1}, k \geq 0} \Xi((\alpha_j)_{1 \leq j \leq p-1}, k) \prod_{j=1}^{p-1} (-r_j)^{\alpha_j} \quad (45)$$

We use here the same scheme as in the first case $p = 2$. We have:

$$\mathcal{L}^p = \left(\frac{1}{4} \partial_\xi^2\right)^p + O_1 \quad (46)$$

thus eq. (44) yields:

$$\partial_\tau \varphi_2 = -r_p \partial_\xi^{2p+1} \varphi_2 + O_2. \quad (47)$$

We assume that this identity is satisfied for all $p \leq p_0$, and consider the r.h.s. member $\Xi_n(\varphi_2, \varphi_4, \ldots, \varphi_{n-2})$ where $\partial_{\tau_0} \varphi_2$ appears for the first time: $n = 2p_0$. Making use of eqs. (47) and (45) in the expression (B53) of $\Xi_n$ shows that the linear terms in $\varphi_2$ vanish from it. This proves that the equation (44) effectively governs the $\tau_0$ evolution of $\varphi_2$. By mathematical induction this statement is valid for any $p$. The equations (44) are the so-called KdV Hierarchy; it is the only compatible set of partial differential equations that has the same homogeneity properties as the sequence of the equations of the perturbative frame. Thus the main term obeys universal equations. All the dependency with regard to the particular physical situation is contained in the scaling coefficients $r_p$ of the higher order time variables $\tau_p$. The study of these coefficients is left for further publication. This concerns the main term only, the higher order terms depend highly on the initial model through
the r.h.s. members $\Xi_p$, and give account for most features particular to the physical situation, especially the lack of integrability.

The approximate solution $u_\varepsilon^{(2)} = \varepsilon^2 \varphi_2(\xi, \tau_1, \tau_2, \ldots, \tau_6)$ defined this way is expected to differ from the exact solution $u$ for some $O(\varepsilon^3)$, when $t$ goes up to $T/\varepsilon^{p+1}$. Despite this paper does not intend nor pretend to prove the convergence, it has to be noticed that this conjecture is reasonable when all higher order terms are bounded with regard to each of the involved time variables. This point yields the matter of next subsection.

4.3 The third order and the linearized KdV Hierarchy

Which approximations can be built from the third order equations

The third equation of the perturbative expansion is eq. (22), written for $l = 6$. It involves the time derivatives $\partial_\tau_1 \varphi_2$, $\partial_\tau_2 \varphi_2$, $\partial_\tau_3 \varphi_2$, $\partial_\tau_1 \varphi_4$, $\partial_\tau_2 \varphi_4$, and $\partial_\tau_3 \varphi_6$. Solving the corresponding step of the perturbative scheme consists thus in building 3 functions: $\varphi_2(\xi, \tau_1, \tau_2, \tau_3)$ of 3 time variables, $\varphi_4(\xi, \tau_1, \tau_2)$ of 2 time variables, and $\varphi_6(\xi, \tau_1)$ of only one time variable. These functions will be built in such a way that they are bounded with regard to each of their time variables. Then the approximate solution $u_\varepsilon^{(2)} = \varepsilon^2 \varphi_2(\xi, \tau_1, \tau_2, \tau_3)\bar{u}_1$ is expected to be close to the exact solution $u$ up to times $t$ with the same order of magnitude as $1/\varepsilon^7$. We give first an heuristic reasoning in order to show that this convergence conjecture is reasonable. The boundedness will be discussed thereafter. The basic idea of the reasoning is that the order of magnitude of the rest is the same as the one of the first non-written term. We assume that the following order is solved. The functions $\varphi_4$ and $\varphi_6$ grow at most linearly with regard to the following time variable, the first non-written one, and $\varphi_8$ evolves in the same way with regard to $\tau_1$. This statement gives account for the existence of secularities. Then, for $t \sim T/\varepsilon^7$, $\varepsilon^4 \varphi_4 \in O(\varepsilon^4 \tau_3) = O(\varepsilon^4)$, and, in the same way, $\varepsilon^5 \varphi_6$ and $\varepsilon^5 \varphi_8 \in O(\varepsilon^5)$. Thus the first terms of the expansion that are left by side are $O(\varepsilon^4)$, and the main term is an approximation within this accuracy (precisely, $u = \varepsilon^2 \varphi_2 \bar{u}_2 + \varepsilon^3 u_3^0 + O(\varepsilon^4)$). An analogous reasoning shows that, for $t \leq T/\varepsilon^5$, $u = u_\varepsilon^{(5)} + O(\varepsilon^6)$, with:

$$u_\varepsilon^{(5)} = \varepsilon^2 \varphi_2(\xi, \tau_1, \tau_2, \tau_3)\bar{u}_1 + \varepsilon^3 u_3^0 + \varepsilon^4 \left( u_1^0 + \varphi_4(\xi, \tau_1, \tau_2)\bar{u}_1 \right) + \varepsilon^5 u_5^0.$$ (48)

It shows also that, for $t \leq T/\varepsilon^3$, $u = u_\varepsilon^{(7)} + O(\varepsilon^8)$. All these statements, although they are not proved, involve the boundedness of the higher order terms, not only with regard to the main time variable $\tau_1$, but also with regard to the higher order time variables, namely the $\tau_2$-dependency of $\varphi_4$, at this point of the expansion.

The linearized KdV hierarchy

The discussion of the boundedness of $\varphi_6$ involves the computation of the linear terms in the r.h.s. member $\Xi_6(\varphi_2, \varphi_4)$ of the linearized KdV equation under consideration. $\Xi_6(\varphi_2, \varphi_4)$ is given by formula (B53). We make use of the first three equations of the hierarchy for $\varphi_2$ (equation (47) written for $p = 1, 2, 3$). For $p = 1$ it is the KdV equation (21) itself, and $\Xi_6$ reads:

$$\Xi_6(\varphi_2, \varphi_4) = -\partial_\tau_2 \varphi_4 + \Xi((2, 0, \ldots, 1) \left( \int_{-\infty}^{\xi} \partial_\tau_1 \right)^2 \partial_\xi \varphi_4 + \Xi((1, 0, \ldots, 3) \left( \int_{-\infty}^{\xi} \partial_\tau_1 \right)^2 \partial_\xi^2 \varphi_4 + \Xi((0, 0, \ldots, 5) \partial_\xi^3 \varphi_4 + O_2 \right).$$ (49)

($O_2$ represents here a function of $\varphi_2$ and $\varphi_4$ without linear terms). The $\tau_1$-evolution of $\varphi_4$ is known, and described by the linearized KdV equation (22), with the r.h.s. member $\Xi_4(\varphi_2)$. Thus we have:

$$\int_{-\infty}^{\xi} \partial_\tau_1 \varphi_4 = -r \partial_\xi^2 \varphi_4 + O_2$$ (50)

Reporting eq. (50) into (49), we get:

$$\Xi_6(\varphi_2, \varphi_4) = \partial_\tau_2 \varphi_4 + r_2 \partial_\xi^2 \varphi_4 + O_2$$ (51)
(Here $O_2$ represents some expression in $\varphi_2, \varphi_4$, their derivatives and primitives, without linear terms.)

The terms involving $\varphi_4$ in the r.h.s. of eq. (51) are \textit{a priori} secular producing. It is shown that they in fact vanish as follows.

Every function $\varphi_n$ can be divided into two parts $\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)}$. The first term $\varphi_n^{(1)}$ is a solution of the homogeneous linearized KdV equation with the given initial data, and $\varphi_n^{(2)}$ is a solution of the inhomogeneous equation, with the r.h.s. member $\Xi_n(\varphi_2, \ldots \varphi_{n-2})$, and zero initial data. For each $p$, the $\tau_p$ dependency of both $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$ is not free. For $\varphi_4^{(2)}$, it is completely determined by the spectral transform, through the explicit resolution formula of the linearized KdV equation [14], while $\varphi_n^{(1)}$ satisfies all the equations of the linearized KdV Hierarchy [17]:

$$\partial_{\tau_p} \varphi_n^{(1)} + (-4)^p \tau_p \partial_\xi D_\varphi \varphi_n^{(1)} = 0$$

In eq. (52), $D_\varphi$ is defined by:

$$D_\varphi \varphi_n^{(1)} = \left( d_1 \mathcal{L}^{p-1} + \mathcal{L} d_1 \mathcal{L}^{p-2} + \cdots \right) \left( \cdots + \mathcal{L}^{p-1} d_1 \right) \frac{q}{6r} \varphi_2 + \mathcal{L}^{p} \varphi_n^{(1)}$$

and:

$$d_1 = \frac{6r}{q} \frac{d\mathcal{L}}{d\varphi_2} (\varphi_n^{(1)}) = -\varphi_n^{(1)} + \frac{1}{2} \int d\xi (\partial_\xi \varphi_n^{(1)})$$

Further, it has been shown in [21] that $\varphi_4^{(2)}$ is not secular producing, while $\varphi_n^{(1)}$ is so. It can be admitted that all linear terms involving $\varphi_n^{(1)}$ also are secular producing, while those involving $\varphi_n^{(2)}$ are not.

**Boundedness at third order**

Eq. (52), written for $n = 4$ and $p = 2$, together with eqs. (53), (54), and (46), yields:

$$\partial_{\tau_2} \varphi_4^{(1)} = -\tau_2 \partial_\xi^2 \varphi_4^{(1)} + O_2$$

and all linear terms vanish from $\Xi_6(\varphi_2, \varphi_4)$, except those involving $\varphi_4^{(2)}$. Because the latter are not secular producing, it is seen that all secular producing terms are removed, and thus $\varphi_6$ is not secular.

The same feature holds at every higher order. According to [14], all nonsecular solutions of the linearized KdV equation (22) are bounded, thus $\varphi_6$ is bounded. This shows that the complete expansion can be solved with all terms bounded with regard to $\tau_1$.

The boundedness of $\varphi_4$ with regard $\tau_2$ rests on completely different grounds. The solutions $\varphi_4$ of the linearized KdV equation can be expanded into a sum and integral of the squared Jost functions $\Psi_k$ [14],[21]. Obviously, its dependency with regard to the higher order time variable $\tau_2$ appears in this expansion through the dependency of the $\Psi_k$ and through those of the coefficients. These coefficients are, apart from factors depending on the Cauchy data, scalar products of the squared Jost function $\Phi_k$ and the initial Cauchy data $\varphi_4|_{t=0}$, or scalar products of the $\Phi_k$ with the r.h.s. member $\Xi_4(\varphi_2)$. Thus all terms in this expansion are functions of $\varphi_2$, algebraically or through the inverse scattering transform. Beside $\varphi_2$, they can depend only on the initial Cauchy data $\varphi_4|_{t=0}$, that is constant with regard to any time variable. Thus the dependency of $\varphi_4$ with regard to $\tau_2$ comes from the one of $\varphi_2$ only. Thus if the regularity properties of the Hirota $\tau$-function ensure that $\varphi_2$ is bounded with regard to $\tau_2$, $\varphi_4$ will also be bounded. Notice that these considerations do not yield a rigorous mathematical proof for the boundedness of $\varphi_4$, for which precise regularity hypothesis, adequate norm definitions, and estimations would have to be given. We have, not properly speaking proved, but show that it is reasonable to admit that $\varphi_4$ is bounded with regard to $\tau_2$. The same reasoning holds, in the same heuristic way, to show that the higher order terms are all bounded regard to the higher order time variables. This ensures the validity of the above mentioned long-time approximation.

**5 Conclusion**

We have studied the higher order terms in the perturbative expansion that describes KdV solitons in ferromagnetic materials. Using various mathematical techniques, we have been able to write down the
equations satisfied by the quantities of any order in this expansion: the form and the coefficients of these equations are explicitly known; in every case, it is the linearized KdV equation, with some r.h.s. that can be explicitly computed by recurrence formulas. This yields approximate value of the solution with an arbitrary accuracy, but on rather small time intervals, due to unbounded or secular solutions. These secular terms are removed, and the approximation is extended to longer propagation times by imposing to the main term to satisfy all equations of the KdV Hierarchy.

This is an application of the multiple times formalism by Kraenkel, Pereira and Manna [3], [5], [6] to ferromagnetism, with numerous improvements, the first of which is that the equations are derived at any order. Second, the homogeneity and parity properties of the r.h.s. members of the linearized KdV equations have been precisely stated and proved. In particular, the considered problem involves nonzero odd order terms in the expansion: we proved that the odd order linearized KdV equations are homogeneous, thus only even order equations remain. This situation is more general than in the previous works, in which these terms vanished without explanation. Third, the approximation obtained at each order is precise. The precision obtained depends on both the number of terms considered in the expansion, and on the order of magnitude of the propagation time. The evolution of the higher order terms with regard to the higher order variables is considered, and not only the dependency of the main term with regard to the higher order time variables, and of the higher order terms with regard to the first order variable. Finally, we show that all terms at order three in the expansion are bounded, in a way that generalizes straightforwardly to any order. All these results needed the introduction of adequate mathematical techniques. Some of them might be of interest on their own.

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Appendices

A Derivation of the KdV equation and its higher orders

A.1 The zero order

For $p = 0$, Eq. (16) writes:

$$B(u_0, u_0) = 0 \quad (A1)$$

that is:

$$\vec{H}_0 = \lambda \vec{M}_0 \quad (A2)$$

where $\lambda$ is a (real) function of $(\xi, \tau)$ to be determined.

For $p = 1$, it writes, after integration and some direct algebra, using eq. (A2):

$$\vec{E}_0 = -\frac{1}{V} R_x (\vec{H}_0 - \alpha \vec{m}) \quad (A3)$$

$$\vec{M}_0 - \vec{m} = -\Gamma (\vec{H}_0 - \alpha \vec{m}) \quad (A4)$$

$$-V \partial_\xi \vec{M}_0 = -\vec{M}_0 \wedge (\vec{H}_1 - \lambda \vec{M}_1) \quad (A5)$$

We have set:

$$\Gamma = 1 + \frac{1}{V^2} R^2_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

with $\gamma = 1 - 1/V^2$. If we assume that the quantity $\mu = 1 + \alpha \gamma$ is not zero, this allows us to express
\( \vec{M}_0 \) as a function of \( \lambda \). Multiplying (dot product) eq. (A5) by \( \vec{M}_0 \), we see that \( ||\vec{M}_0||^2 \) is a constant. Then, computing this quantity, we find that \( \lambda \) satisfies a fourth degree polynomial equation with constant coefficients. Being continuous, \( \lambda \) is thus a constant, and we have proved that \( u_0 \) is a constant (equal to \( \vec{m} \)).

Then eqs. (A3)-(A5) reduce to the compatibility condition:

\[
\vec{m} \wedge (\vec{H}_1 - \alpha \vec{M}_1) = \vec{0}
\]

A.2 Resolution for a given step

For every \( p \geq 2 \), eq. (16) reads (\( n = p - 1 \geq 1 \)):

\[
-V \partial_x \vec{E}_n + \sum_{j \geq 1} \partial_{r_j} \vec{E}_{n-2j} = R_x \partial_x \vec{H}_n \tag{A6}
\]

\[
-V \partial_x \vec{H}_n + \sum_{j \geq 1} \partial_{r_j} \vec{H}_{n-2j} = -R_x \partial_x \vec{E}_n + \sum_{j+k=n+1} \vec{M}_j \wedge \vec{H}_k \tag{A7}
\]

\[
-V \partial_x \vec{M}_n + \sum_{j \geq 1} \partial_{r_j} \vec{M}_{n-2j} = -\sum_{j+k=n+1} \vec{M}_j \wedge \vec{H}_k \tag{A8}
\]

First we compute \( \vec{E}_n \) and \( \vec{M}_n \) in terms of \( \vec{H}_n \) in eqs. (A6) and (A7), and obtain eqs. (19) and (20). In these equations, we use the following definition:

\[
\vec{m}_I \left( \begin{array}{c} \vec{E} \\ \vec{H} \\ \vec{M} \end{array} \right) = \frac{1}{V} (\vec{H} + \vec{M}) + \frac{1}{V^2} R_x \vec{E} \tag{A9}
\]

Then we report this result in eq. (A8), written at preceding order (\( n - 1 \)). This yields the equation (18), where the linear operator \( L \) is defined by:

\[
L \vec{H} = \vec{m} \wedge [(1 + \alpha \Gamma) \vec{H}] \quad \text{for each} \quad \vec{H} \tag{A10}
\]

and:

\[
\vec{W}_{n-1} = \alpha \vec{m} + \sum_{j \geq 1} \int_{-\infty}^{\xi} \partial_{r_j} \vec{m}_I(u_{n-2j}) + V \partial_x \vec{M}_{n-1} - \sum_{j \geq 1} \vec{M}_j \wedge \vec{H}_{n-j} \tag{A11}
\]

The system yielded by eqs. (19-20-18) for each \( n \geq 1 \) is equivalent to the system yielded by eq. (16) for each \( p \geq 1 \) (for a given \( u_0 \)). The vector equation \( L \vec{H} = \vec{W} \), for any given r.h.s. \( \vec{W} \), writes

\[
L \vec{H} = \left( \begin{array}{c} \mu m_x H_z \\ -\mu m_x H_z \\ \mu m_x H_y - (1 + \alpha)m_t H_x \end{array} \right) = \vec{W} \tag{A12}
\]

The solvability condition is:

\[
\vec{m} \cdot \vec{W} = 0 \tag{A13}
\]

If this is satisfied, the solution \( \vec{H} \) writes:

\[
\vec{H} = \varphi \vec{h}_1 + L^{-1} \vec{W} \tag{A14}
\]

where \( \varphi \) is any real quantity, \( \vec{h}_1 \) the vector:

\[
\vec{h}_1 = \left( \begin{array}{c} \mu m_x \\ (1 + \alpha)m_t \\ 0 \end{array} \right) \tag{A15}
\]

\( L^{-1} \) is a partial inverse of \( L \), i.e. the matrix:

\[
L^{-1} = \frac{1}{\mu m_x m_t} \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & m_t \\ m_x & 0 & 0 \end{array} \right) \tag{A16}
\]

The present computation is a particular case of the general theory of [17]. Notice that the general theory applies because the dimension of the kernel of \( L \) is 1.

The whole perturbative scheme can be now written under the following form: \( u_0 \) is given and for each \( n \geq 1 \), the condition:

\[
\vec{W}_{n-1} \cdot \vec{m} = 0 \tag{A17}
\]

is satisfied, and \( u_n \) is given by:

\[
\vec{H}_n = \varphi_n \vec{h}_1 + L^{-1} \vec{W}_{n-1} \tag{A18}
\]

and the formulas (19-20), that give \( \vec{E}_n \) and \( \vec{M}_n \cdot \varphi_n \) is a function to be determined.

This can be written in the condensed form (for each \( n \geq 2 \)):

\[
u_n = \varphi_n \vec{u}_1 + u_n^0 \tag{A19}\]
\( u_n^0 = u_n^0(\varphi_2, \ldots, \varphi_{n-1}) \) expresses as an explicit function of the previous order terms, according to:

\[
 u_n^0 = S\tilde{W}_{n-1} + \sum_{j=1}^{\infty} \int_{-\infty}^{\xi} \partial_j u_I(u_{n-2j}) \tag{A20}
\]

where \( S \) is the \( 9 \times 3 \) matrix:

\[
 S = TL^{-1} \quad \text{with} \quad T = \begin{pmatrix} -\frac{1}{\nu} R_x \\ I \\ -\Gamma \end{pmatrix} \tag{A21}
\]

(\( I \) is the three-dimensional unity matrix), \( \tilde{u}_1 = \begin{pmatrix} \tilde{e}_1 \\ \tilde{h}_1 \end{pmatrix} = T\tilde{h}_1 \), and \( u_I \) is the linear operator in \( \mathbb{R}^9 \) defined by:

\[
u \begin{pmatrix} E \\ \tilde{E} \\ \tilde{M} \end{pmatrix} = \begin{pmatrix} \frac{1}{\nu} \tilde{E} \\ 0 \\ \tilde{m}_I \begin{pmatrix} \tilde{E} \\ \tilde{H} \\ \tilde{M} \end{pmatrix} \end{pmatrix} \tag{A22}
\]

Together with condition (A17), the two recurrence formulas (A18) and (A11) (which defines \( \tilde{W}_n \)) describe completely the perturbation scheme, for a given \( u_0 = \tilde{m} \).

**A.3 The evolution equations**

Now we will see that condition (A17), for a convenient choice of the velocity \( V \), is an evolution equation for \( \varphi_{n-2} \). To this aim, we introduce a notation inspired from the Landau notation \( O(x^n) \) or \( o(x^n) \) for the error in a polynomial approximation. For each integer \( k \), we will write \( T_k \) for any quantity that expresses in terms of the functions \( \varphi_1, \varphi_2, \ldots \), up to \( \varphi_k \), their derivatives and primitives. It is very easy to show by induction that:

\[
u n = \varphi_n \tilde{u}_1 + T_{n-1} \tag{A23}
\]

for each \( n \geq 1 \). (This means that the only term containing \( \varphi_n \) in the complete expression of \( u_n \) is \( \varphi_n \tilde{u}_1 \).) Expression (A23) is reported into the recurrence formula (A11) for \( \tilde{W}_n \), and gives:

\[
u \tilde{m} \cdot \tilde{W}_n = V\tilde{m} \cdot \tilde{m}_1 \partial_\xi \varphi_n - 2\tilde{m} \cdot [\tilde{m}_1 \wedge \tilde{h}_1] \varphi_1 \varphi_n + T_{n-1} \tag{A24}
\]

Let us consider the first nonzero \( \varphi_n \); we call it \( \varphi_{n_0} \). Reporting expression (A24) in the equation \( \tilde{m} \cdot \tilde{W}_n = 0 \), we get, if \( n_0 = 1 \):

\[
u V\tilde{m} \cdot \tilde{m}_1 \partial_\xi \varphi_1 - 2\tilde{m} \cdot [\tilde{m}_1 \wedge \tilde{h}_1] \varphi_1^2 = 0 \tag{A25}
\]

and else:

\[
u V\tilde{m} \cdot \tilde{m}_1 \partial_\xi \varphi_{n_0} = 0 \tag{A26}
\]

Neither eq. (A25) nor eq. (A26) admit a nonzero bounded solution vanishing at infinity if \( \tilde{m} \cdot \tilde{m}_1 \neq 0 \). The condition \( \tilde{m} \cdot \tilde{m}_1 = 0 \) must thus be satisfied. It writes:

\[
u \mu m_x^2 + \gamma(1 + \alpha)m_t^2 = 0 \tag{A27}
\]

and can be solved in order to give the velocity \( V \):

\[
u V = \pm \sqrt{\frac{\alpha + \sin^2 \theta}{\alpha + 1}} \tag{A28}
\]

The angle \( \theta \) is the angle between the propagation direction and the constant field, so that:

\[
u \tilde{m} = \begin{pmatrix} m \cos \theta \\ m \sin \theta \\ 0 \end{pmatrix} \tag{A29}
\]

Notice that, as above for the definition of \( L^{-1} \), the product \( m_x m_t \) is assumed to be nonzero, i.e. \( \theta \) differs from 0, \( \pm \pi/2 \), and \( \pi \).

The sign of \( V \) in (A28) corresponds to a free choice of the propagation direction. We choose \( V \) positive. The coefficient \( \tilde{m} \cdot [\tilde{m}_1 \wedge \tilde{h}_1] \) in eq. (A25) vanishes, because the three vectors \( \tilde{m}, \tilde{m}_1 \) and \( \tilde{h}_1 \) belong to the \( xy \) plane (designated by II in subsection 3.2). That this can be proved without computation is interesting, because this kind of reasoning will be useful at higher orders, where direct computation becomes awfully complicated. Thus \( \varphi_{n_0} \) vanishes from the equation \( \tilde{m} \cdot \tilde{W}_n = 0 \). We pursue the computation:

\[
u u_2 = \varphi_2 \tilde{u}_1 + \partial_\xi \varphi_1 \tilde{u}_2 - S(\tilde{m}_1 \wedge \tilde{h}_1) \varphi_1^2 \tag{A30}
\]

and for each \( n \geq 3 \):

\[
u u_n = \varphi_n \tilde{u}_1 + \partial_\xi \varphi_{n-1} \tilde{u}_2 - 2S(\tilde{m}_1 \wedge \tilde{h}_1) \varphi_1 \varphi_{n-1} + T_{n-2} \tag{A31}
\]

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(The difference between eqs. (A30) and (A31) is the coefficient 2 before the nonlinear term. Note that $\mathcal{T}_{n-2}$ is exactly zero for $n = 2$). We define vector coefficients $\tilde{u}_n$ by:

$$
\tilde{u}_n = \left( \begin{array}{c}
\bar{e}_n \\
\bar{h}_n \\
\bar{m}_n
\end{array} \right) = S(V\bar{m}_{n-1}) \quad (A32)
$$

for each $n$. We write the operator $S$ as $S = \left( \begin{array}{c}
\bar{S}_c \\
\bar{S}_h \\
\bar{S}_m
\end{array} \right)$

and compute:

$$
\bar{m} \cdot \bar{W}_2 = V\bar{m} \cdot \bar{m}_2 \partial^3 \varphi_1 - V\bar{m} \cdot \bar{S}_m(\bar{m}_1 \wedge \bar{h}_1) \partial \varphi_1 \bar{m}_1 \wedge \bar{m}_2 \wedge \bar{h}_1) \varphi_1 \bar{m}_1 \wedge \bar{m}_2 \wedge \bar{h}_1) + \bar{S}_m(\bar{m}_1 \wedge \bar{h}_1) \varphi_1
\quad (A33)
$$

Many coefficients in expression (A33) vanish, and this can be found with very few explicit computation. We called $\Pi$ the $xy$ plane, let us call $\Delta$ the $z$ axis. We see easily that $L^{-1}$ maps $\Pi$ into $\Delta$ and reciprocally, while $\Gamma$ does not modify the belonging of a vector to $\Pi$ or $\Delta$. Thus $S_m = -\Gamma L^{-1}$ and $S_h = L^{-1}$ both behave in the same way as $L^{-1}$. Furthermore, the outer product of two vectors of $\Pi$ belongs to $\Delta$. Our starting point is that both $\bar{m}_1$ and $\bar{h}_1$ belong to $\Pi$. Thus $\bar{m}_1 \wedge \bar{h}_1$ belongs to $\Delta$, then $S_h(\bar{m}_1 \wedge \bar{h}_1)$ and $S_m(\bar{m}_1 \wedge \bar{h}_1)$ belong to $\Pi$, and $\bar{m}_1 \wedge \bar{S}_h(\bar{m}_1 \wedge \bar{h}_1)$ and $\bar{m}_1 \wedge \bar{S}_m(\bar{m}_1 \wedge \bar{h}_1) \bar{h}_1$ belong to $\Delta$. Thus $\bar{m} \cdot [\bar{m}_1 \wedge \bar{S}_h(\bar{m}_1 \wedge \bar{h}_1) + \bar{S}_m(\bar{m}_1 \wedge \bar{h}_1) \bar{h}_1] = 0$.

The coefficient of $\varphi_1 \partial \varphi_1$ must be computed explicitly, and we find that:

$$
\bar{m} \cdot \bar{W}_2 = 3V(1 - \gamma)m^2 \varphi_1 \partial \varphi_1 \quad (A34)
$$

The condition $\bar{m} \cdot \bar{W}_2 = 0$ yields thus $\varphi_1 = 0$. All the terms in $\varphi_{n-1}$ vanish then from $\bar{m} \cdot \bar{W}_n$, and we must go further in the computation to make appear $\varphi_{n-2}$. We find that:

$$
u_n = \varphi_n \bar{u}_1 + \partial \varphi_3 \bar{u}_2 + \partial \varphi_3 \bar{u}_3 - S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 + \Phi(\bar{u}_1) \int_{-\infty}^{\xi} \partial \varphi_2 \quad (A35)
$$

and, for each $n \geq 5$:

$$
u_n = \varphi_n \bar{u}_1 + \partial \varphi_n \bar{u}_2 + \partial \varphi_n \bar{u}_3 - 2S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 \varphi_n - 2S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 \varphi_n - 2S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 \varphi_n + \Phi(\bar{u}_1) \int_{-\infty}^{\xi} \partial \varphi_2 + \mathcal{T}_{n-3} \quad (A36)
$$

with:

$$
\Phi(u) = S(\alpha \bar{m} \wedge \bar{m}_1(u)) + u_1(u) \quad (A37)
$$

Thus:

$$
\bar{m} \cdot \bar{W}_n = \Lambda \varphi_n - 2q \varphi_n \partial \varphi_n + r \partial^3 \varphi_n \quad (A38)
$$

and for $n \geq 5$:

$$
\bar{m} \cdot \bar{W}_n = \Lambda \varphi_n - 2q \varphi_n \partial \varphi_n + r \partial^3 \varphi_n - 2S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 \varphi_n - 2S(\bar{m}_1 \wedge \bar{h}_1) \varphi_2 \varphi_n + \Phi(\bar{u}_1) \int_{-\infty}^{\xi} \partial \varphi_2 + \mathcal{T}_{n-3} \quad (A39)
$$

The equation $\bar{m} \cdot \bar{W}_n = 0$ is thus an evolution equation for $\varphi_{n-2}$ (eqs. (21) and (22)).

The coefficients $q, r$, and $\Lambda$ are defined by:

$$
\Lambda = V \bar{m} \cdot \bar{m}_1(\bar{u}_1) \quad (A40)
$$

$$
q \Lambda = -2V \bar{m} \cdot \bar{S}_m(\bar{m}_1 \wedge \bar{h}_1) \quad (A41)
$$

$$
r \Lambda = V \bar{m} \cdot \bar{m}_3 \quad (A42)
$$

Explicit computation gives:

$$
q = -\frac{3}{2} \gamma m \nu \quad (A43)
$$

$$
r = -\frac{\gamma^2 \nu^5}{2(1 + \alpha)m^2} \quad (A44)
$$

We find again the expressions that have been obtained in ref. [10].

B Computation of the linear terms in the r.h.s. of the equations of the perturbative expansion

The r.h.s. member $\Xi_l(\varphi_2, \varphi_3, \ldots, \varphi_{l-1})$ of the linearized KdV equation (22) is defined by (for each

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\( l \geq 3 \):
\[
\Xi_l(\varphi_2, \varphi_3, \ldots \varphi_{l-1}) = -\frac{1}{\Lambda} \vec{m} \cdot \vec{W}_{l+2} + \partial_{\tau_1} \varphi_l + q \partial_{\xi}(\varphi_2 \varphi_l) + r \partial_{\xi}^3 \varphi_l
\]
(B45)

We will write \( O_2 \) for any polynomial expression in the \( \varphi_j \) and their derivatives and primitives, that doesn’t contain any linear term. This is analogous to the usual notation \( O(\varphi_n^2) \), except that we don’t assume that the \( \varphi_j \) are small in any way.

Notice that the use of the KdV equation satisfied by \( \varphi_2 \), and so on, could change the linear terms in \( \Xi_l(\varphi_2, \varphi_3, \ldots \varphi_{l-1}) \), thus in order to define the linear part of \( \Xi_l(\varphi_2, \varphi_3, \ldots \varphi_{l-1}) \) in a unique way, we assume that no use of these properties has been done. We show elsewhere that \( \varphi_1 \) is necessary zero, and even that, for zero initial conditions at these orders, all \( \varphi_j \) with an odd value of \( j \) are zero. But it is more convenient for the present formal computation to forget this feature. It is consistent with the requirement that no use of the equations has been done: we compute formally the r.h.s. without solving the equations in any way.

While we know the homogeneity properties of \( u_n \) and \( W_n \) (see section 3; this homogeneity is relative to the “degree” \( d \) defined by expression (25)), we can write an \textit{a priori} formula for the linear part of \( u_n \):
\[
u_n = \sum_{(\alpha_j)_{j \geq 1}, k \geq 0, l \geq 1} \tilde{u}( (\alpha_j)_{j \geq 1}, k, l ) \left( \sum_{j \geq 1} 2 j \alpha_j \right)^{k+l-n} \prod_{j \geq 1} \left( \int_{-\infty}^{\xi} \partial_{\tau_j} \right) \alpha_j \partial_{\xi}^j \varphi_l + O_2
\]
(B46)

where the 9-components coefficients:
\[
\tilde{u}( (\alpha_j)_{j \geq 1}, k, l ) = \begin{pmatrix}
\vec{e}( (\alpha_j)_{j \geq 1}, k, l ) \\
\vec{h}( (\alpha_j)_{j \geq 1}, k, l ) \\
\vec{m}( (\alpha_j)_{j \geq 1}, k, l )
\end{pmatrix}
\]

have to be determined. Reporting formula (B46) into the recurrence formula (A19) for \( u_n \) yields new recurrence formulas, that allow to compute these coefficients. This shows by induction that formula (B46) is valid for any value of the integer \( n \), and that the values found for the coefficients are valid.

These recurrence formulas read as follows, for all \( l \geq 1 \):
\[
\partial(0), 0, l) = \tilde{u}_1
\]
(B47)

For all \( k \) and \( l \geq 1 \):
\[
u((\alpha_j)_{j \geq 1}, 0, l) = S(V\vec{m} ((\alpha_j)_{j \geq 1}, k - 1, l))
\]
(B48)

\( S \) is defined by eq. (A21). For all \( (\alpha_j)_{j \geq 1} \neq (0) \) and \( l \geq 1 \):
\[
u((\alpha_j)_{j \geq 1}, k, l) = S(V\vec{m} ((\alpha_j)_{j \geq 1}, k - 1, l)) + \sum_{i \geq 1} \Phi(\tilde{u}((\alpha_j - \delta_{i,j})_{j \geq 1}, 0, l)) - S(V\vec{m} ((\alpha_j - \delta_{i,j})_{j \geq 1}, k - 1, l))
\]
(B49)

\( \delta_{i,j} \) is the Kronecker symbol and \( \Phi \) is defined by eq. (A37). For all \( (\alpha_j)_{j \geq 1} \neq (0) \), \( k, l \geq 0 \):
\[
u((\alpha_j)_{j \geq 1}, k, l) = S(V\vec{m} ((\alpha_j)_{j \geq 1}, k - 1, l)) + \sum_{i \geq 1} \Phi(\tilde{u}((\alpha_j - \delta_{i,j})_{j \geq 1}, 0, l)) - S(V\vec{m} ((\alpha_j - \delta_{i,j})_{j \geq 1}, k - 1, l))
\]
(B50)

Formulas (B47) to (B50) define \( \tilde{u}((\alpha_j)_{j \geq 1}, k, l) \). We see that this quantity doesn’t depend on \( l \) (thus we will write more simply \( \tilde{u}((\alpha_j)_{j \geq 1}, k) \)), and that, for each \( k \geq 0 \):
\[
\tilde{u}(0), k) = \tilde{u}_{k+1}
\]
(B51)

as it is defined by eq. (A32). The obtained formulas are reported in the definition (A11) of \( \vec{W}_n \) to yield:
\[
\vec{m} \cdot \vec{W}_n = \sum_{(\alpha_j)_{j \geq 1}, k \geq 0, l \geq 1} \left( \sum_{j \geq 1} 2 j \alpha_j \right)^{k+l-n} \prod_{j \geq 1} \left( \int_{-\infty}^{\xi} \partial_{\tau_j} \right) \alpha_j \times \vec{m}( (\alpha_j)_{j \geq 1}, k - 1) + O_2
\]
(B52)
Thus the r.h.s. $\Xi_n$ writes:

$$
\Xi_n(\varphi_2, \varphi_3, \ldots, \varphi_{n-1}) = \sum_{(\alpha_j)_{j\geq 1}, k\geq 0} \Xi((\alpha_j)_{j\geq 1}, k) \times \\
(\sum_{j\geq 1} 2\delta_{\alpha_j}) + k + l = n + 3
$$

(B53)

\[ \times \prod_{j\geq 1} \left( \int_{-\infty}^{\xi} \partial_{\tau_j} \right) \delta_{\xi} \varphi_l + O_2 \]

with:

$$
\Xi((\alpha_j)_{j\geq 1}, k) = -\frac{1}{\Lambda} \left[ V \tilde{m} \cdot \tilde{m} ((\alpha_j)_{j\geq 1}, k - 1) \\
- \sum_{i\geq 1} \tilde{m} \cdot \tilde{m} ((\alpha_j - \delta_{i,j})_{j\geq 1}, k - 1) \right] \quad (B54)
$$

$\tilde{m} ((\alpha_j)_{j\geq 1}, k)$ is defined by the recurrence formulas (B47) to (B50). Note the remarkable feature that the coefficient $\Xi((\alpha_j)_{j\geq 1}, k)$ doesn’t depend on $n$. In some important particular cases, it can be computed explicitly. We seek for the term in which $\partial_{\tau_j} \varphi_l$ appears with the largest value of the index $j$, in a given $\Xi_n$. Because $n$ is necessary even, we write it $= 2p$. Due to the homogeneity properties of $\Xi_n(\varphi_1, \ldots)$ (see section 3), we see that it is the term proportional to $\partial_{\tau_j} \varphi_2$. The coefficient is:

$$
\Xi((\delta, p), 1) = -1 \quad (B55)
$$

C Computation of the $\delta$-degree of $u_k$. Proof of relation (31)

The relation to be proved reads:

$$
\delta(u_k) \equiv k \quad [2] \quad (C56)
$$

It is easily checked for $k = 1 : u_1 = \varphi_1 \tilde{u}_1$, thus $\delta(u_1) = 1$. We check that, if the expressions are homogeneous, so that the degrees exist:

$$
\delta_{\Pi}(\tilde{m} \wedge \tilde{h}) \equiv \delta_{\Pi}(\tilde{h}) + \delta_{\Pi}(\tilde{m}) + 1 \quad [2]
$$

Thus:

$$
\delta(\tilde{m} \wedge \tilde{h}) \equiv \delta(\tilde{h}) + \delta(\tilde{m}) + 1 \quad [2] \quad (C57)
$$

It is also easy to check that the operator $\tilde{m}_l$ involved by the recurrence formula (20) satisfies:

$$
\delta(\tilde{m}_l(u)) \equiv \delta(u) \quad [2] \quad (C58)
$$

From (C57) and (C58) is seen that, if $\delta(u_k) \equiv k \quad [2]$ for each $k < n$, then:

$$
\delta(\tilde{W}_n) \equiv n \quad [2] \quad (C59)
$$

Using the fact that the operator $S$ that appears in the recurrence formula (A19) for $u_n$ satisfies:

$$
\delta(S\tilde{h}) \equiv \delta(\tilde{h}) + 1 \quad [2]
$$

we get the $\delta$-degree of $u_n$:

$$
\delta(u_n) \equiv \delta(\tilde{W}_{n-1}) + 1 \equiv n \quad [2] \quad (C60)
$$

which completes the proof.

References


