Dissipative solitons: The finite bandwidth of gain as a viscous friction

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We consider the effect on the motion of a dissipative soliton of the diffusion term in the quintic complex Ginzburg-Landau (CGL) equation, which accounts for the finite bandwidth of the gain when this model describes light pulse evolution in a fiber laser. We show analytically that, if the velocity is small enough, this effect can be modeled by a viscous friction force acting on the soliton. Numerical resolution of the CGL equation shows that this analytical approximation is valid with good accuracy in the case of anomalous dispersion.

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I. INTRODUCTION

Interactions of dissipative solitons, and their collective behavior, are a matter of great interest, both from a fundamental point of view since solitons may be considered to be a model of elementary particles and in the scope of applications in laser physics. Experiments performed with a fiber laser have shown collective behaviors of large collections of solitons comparable to states of matter [1–3]. From the point of view of theoretical physics, an especially relevant frame for the study of dissipative solitons and their interactions is the complex Ginzburg-Landau (CGL) equation. The mathematical properties of this nonlinear partial differential equation have motivated many studies (see, e.g., the review in [4]), and it is closely related to the dynamics of mode-locked lasers. Recall that Haus’s master equation for mode locking [5] is nothing more than the stationary version for the CGL equation. The CGL equation was derived for a fiber laser mode locked by nonlinear rotation of polarization in both its cubic [6] and quintic [7,8] versions and also for a figure-eight laser (cubic CGL) [9].

Although binary interactions were considered theoretically a long time ago [10,11], a simplified model allowing us to account for them as forces between effective particles is still missing, and consequently, the description of large assemblies of solitons still requires extensive numerical computations. Several studies have been devoted to the reduction of the CGL equation to a set of ordinary differential equations, especially the collective-variables approach [12,13]; however, it still fails to allow a convenient description of interactions.

Among the reasons for this obstruction, an essential point is that the dissipative solitons cannot move freely. In the frame of the complex Ginzburg-Landau equation, the term which accounts for the limited bandwidth of the gain breaks the translation invariance and prevents soliton motion. Although moving solitons have been evidenced [14], their motion is due to the asymmetry of their structure and is completely defined by it. They have a fixed nonzero velocity exactly in the same way the symmetrical pulses have a fixed zero velocity. Likewise, it has been shown that some soliton motion can be recovered by adding a continuous component [15]; however, this procedure does not restore translation invariance, and the scheme of Newton’s mechanic still cannot be used to generate a simple model of quasiparticles for the collection of solitons.

In the present paper, we show that the braking due to the limited-bandwidth term is equivalent to a viscous friction, provided that the velocity is small enough, which is the case in experiments. In Sec. II, we derive analytically an expression of this friction force. In Sec. III, we show numerically that the analytical approximation actually describes the pulse motion.

II. ANALYTICAL EXPRESSION OF THE VISCOUS FRICTION

We consider the CGL equation in its standard dimensionless form as

\[ u_z = \delta u + \left( \beta + i \frac{D}{2} \right) u_{tt} + (\varepsilon + i) |u|^2 + (\mu + i \nu) |u|^4. \]  

(1)

The coefficients \( \beta, D, \varepsilon, \mu, \) and \( \nu \) have their standard meaning; that is, \( \beta \) measures the narrowness of the gain spectrum, \( D \) is the dispersion, \( \varepsilon \) and \( \mu \) are the cubic and quintic nonlinear gains (absorption if \( \varepsilon, \mu < 0 \)), and \( \nu \) is the fourth-order nonlinear index. Let us consider some solution \( u_0(t,z) \) of Eq. (1) with \( \beta = 0 \). Then

\[ u = u_0(t-T, z) e^{i(\omega t-kz)} \]  

(2)

is a solution of (1) with \( \beta = 0 \) provided that

\[ T = V z, \quad \omega = \frac{V}{D}, \quad k = \frac{V^2}{2D}. \]  

(3)

Consider some (small) nonzero \( \beta \). If \( u \) is a soliton solution, \( M = \int_{-\infty}^{+\infty} |u|^2 dt \) is the energy of the soliton and can play the role of its mass, and \( T = \int_{-\infty}^{+\infty} t|u|^2 dt/M \) is the position of its center of mass. We assume that \( u \) is a moving stationary solution, typically of the form (2), and consequently, its mass is constant, \( dM/dz = 0 \).

The velocity of the pulse is then

\[ \frac{dT}{dz} = \frac{1}{M} \int t(u, u^{*} + \text{c.c.}) dt \]  

(4)

(c.c. stands for complex conjugate, and the subscript indicates the derivative). Using the CGL equation (1), we transform it into

\[ \frac{dT}{dz} = \frac{1}{M} \left( I_1 + i \frac{D}{2} I_2 + \beta I_3 \right) dt, \]  

(5)
where we have set

\[ I_1 = 2 \int (\delta^2 |u|^2 + \varepsilon |u|^4 + \mu |u|^6)dt, \quad (6) \]

\[ I_2 = \int (u_t u^* - \text{c.c.})dt, \quad (7) \]

\[ I_3 = \int (u_{tt} u^* + \text{c.c.})dt. \quad (8) \]

We assume that \( u_0 \) represents a symmetrical pulse, centered at \( t = T \), and consequently, the function \( u_0(t') \), with \( t' = t - T \), is even. Using the expression (2) of \( u \) and this assumption reduces (6) to

\[ I_1 = 2T \int (\delta^2 |u_0|^2 + \varepsilon |u_0|^4 + \mu |u_0|^6)dt'. \quad (9) \]

Using (2) in (7) after an integration by part gives

\[ I_2 = -2i\omega \int t'|u_0|^2dt'. \quad (10) \]

Equation (8), taking into account the parity of \( u_0 \), is reduced in the same way to

\[ I_3 = -2T \int |u_0|^2 dt'. \quad (11) \]

The derivative

\[ \frac{dM}{dz} = \int (u_t u^* + \text{c.c.})dt \quad (12) \]

of the mass \( M \) is computed, using the integrals \( \int (u_t u^* - \text{c.c.})dt = 0 \) and \( \int (u_{tt} u^* + \text{c.c.})dt = -2 \int |u_0|^2 dt' \), as

\[
\frac{dM}{dz} = 2\delta \int |u_0|^2 dt' - 2\beta \int |u_0|^2 dt' + \varepsilon \int |u_0|^4 dt' + \mu \int |u_0|^6 dt' = 0 \quad (13)
\]

since the mass is assumed to be constant. Using (13) to simplify (5) gives

\[ \frac{dT}{dz} = D\omega, \quad (14) \]

in accordance with (3).

Then we computed the acceleration \( d^2T/dz^2 \). Since \( M \) is constant and, according to (13), \((I_1 + \beta I_3)\) also is constant, the acceleration reduces to

\[ \frac{d^2 T}{dz^2} = \frac{iD}{2M} \frac{dI_2}{dz}. \quad (15) \]

Integration by parts reduces \( dI_2/dz \) to

\[ \frac{dI_2}{dz} = 2 \int (u_t u^* - \text{c.c.})dt. \quad (16) \]

We make use of (1), (2), and the assumption that \( u_0 \) is even and compute the integrals

\[ \int (u_t u - \text{c.c.})dt = -2i\omega \int |u_0|^2 dt', \quad (17) \]

\[ \int (u_{tt} u |u|^2 - \text{c.c.})dt = -2i\omega \int |u_0|^4 dt'. \quad (18) \]

Finally, replacing \( \omega \) with \( V/D \), we obtain the expression of the force \( F = M\frac{d^2T}{dz^2} \) as

\[ F = -4\beta \int |u_0|^2 dt' V - \frac{2\mu}{D} \int |u_0|^2 dt' V^3. \quad (24) \]

The expression of this force contains a term proportional to \( V \), which corresponds to a (linear) viscous friction, and a second term proportional to \( V^3 \), which is a nonlinear correction to the former one. Since the computation assumes (2), i.e., that the motion in the presence of friction does not modify the profile of the pulse, both the coefficient \( \beta \) and the speed \( V \) must remain small. Hence we can expect that the nonlinear term will be negligible in most cases where (24) will be valid.

For small values of \( V \), the cubic term can be neglected; the equation of motion is thus

\[
M\frac{dV}{dz} = F = -4\beta \int |u_0|^2 dt' V, \quad (25)
\]

and consequently, the velocity evolves as \( V(z) = V(0)e^{-\lambda z} \) with the decay rate

\[ \lambda = \frac{4\beta}{M} \int |u_0|^2 dt'. \quad (26) \]

### III. Numerical validation of the approximation

The CGL equation (1) is solved numerically using a standard fourth-order Runge-Kutta scheme in the spectral domain and inverse and direct fast Fourier transforms for the computation of the nonlinear term. We use the set of parameters \( D = 1 \) (anomalous dispersion), \( \delta = -0.03, \varepsilon = 0.4, \nu = 0, \mu = -0.5 \) and vary \( \beta \). The initial data are a steady-state soliton \( u_0 \) of the same equation obtained by running the same code on a long propagation distance multiplied by a phase factor \( e^{i\omega t} \). The initial frequency shift \( \omega \) induces an initial velocity \( V_0 = D\omega \). An example of calculation, with rather small initial velocity \( V_0 = 0.7 \) and gain-bandwidth coefficient \( \beta = 0.004 \), is shown in Fig. 1. The braking of the pulse appears clearly. We compute the mass \( M \), the constant \( \lambda \) defined by (26) using the numerical steady state \( u_0 \), and Simpson’s rule. Then the solution of the approximate equation (25), \( T = T(0) + (V_0/\lambda)(1 - e^{-\lambda z}) \), is also plotted in Fig. 1, in good agreement with the numerical solution.
FIG. 1. The braking of a dissipative soliton with some initial velocity by the finite-gain-bandwidth term. The parameters are $\omega = 0.7, \beta = 0.004$. The white line is the approximate analytical solution deduced from Eq. (25).

To check more accurately the validity of the above approximate analysis, we first see if the mass $M$ is conserved. It is plotted against the propagation distance $z$ on Fig. 2. It is seen that $M$ quickly relaxes to its value at rest, after some transient where it is smaller. After this transient, $M$ is conserved with good accuracy. Then the trajectory of the pulse $T(z)$ is computed using quadratic interpolation of the numerical data; then its velocity $V = dT/dz$ and acceleration $\gamma = d^2T/dz^2$ are calculated using three-point finite differences. $V$ decreases exponentially as expected, as can be seen from Fig. 3, and so does $\gamma$. The decay rate $\lambda$ has been computed using a least-squares fit, in the range $\beta = 2 \times 10^{-4}$ to 0.0124, starting from $\omega = 2$, showing that $\lambda$ is accurately proportional to $\beta$ in accordance with Eq. (26). The force $F$ is computed using the analytic expression (24), where the integrals are computed using Simpson’s rule. It is then compared with $M\gamma$; a typical example of the obtained curves is shown in Fig. 3. There is quantitative agreement between the analytic expression (24) of the braking force and the numerically computed value of acceleration $\gamma$ (multiplied by mass $M$), with accuracy of about 2%, as soon as the velocity $V$ is small enough. More precisely, the relative difference between $F$ and $M\gamma$ goes below 0.1 as $V$ goes below 0.26 (threshold values ranging from 0.25 to 0.29 are obtained for values of $\beta$ varying from 0.004 to 0.025).

For velocities larger than 0.26, the pulse also slows down due to the braking force, but the approximation of it provided by Eq. (24) is not valid. If the initial velocity is further increased, the pulse profile is perturbed, and the pulse may vanish eventually. The vanishing occurs above some limiting value of the initial frequency shift $\omega$, which can be evaluated from numerical data. It is plotted against the gain bandwidth coefficient $\beta$ in Fig. 4. Obviously, the maximal value of $\omega$ can be quite large for very small values of $\beta$ and decreases as $\beta$ increases, tending to stabilize to a value roughly equal to 1.5.

In Fig. 2, the mass $M$ is plotted for two values of the initial frequency shift $\omega$, one of which ($\omega = 1.495$) is close to the limit above which the pulse cannot move and is destroyed. The vanishing occurs above some limiting value of the initial frequency shift $\omega$, which can be evaluated from numerical data. It is plotted against the gain bandwidth coefficient $\beta$ in Fig. 4. Obviously, the maximal value of $\omega$ can be quite large for very small values of $\beta$ and decreases as $\beta$ increases, tending to stabilize to a value roughly equal to 1.5.

In Fig. 2, the mass $M$ is plotted for two values of the initial frequency shift $\omega$, one of which ($\omega = 1.495$) is close to the limit above which the pulse cannot move and is destroyed. The corresponding velocities are given for comparison. At the
beginning of the process, the velocity is large, and the mass decreases. As the velocity goes down to zero, the mass comes back to its value at rest. We see that, when $V$ goes below 0.26, i.e., when the accuracy $\Delta F/F$ of the analytical formula (24) goes below 0.1, we can consider that the mass $M$ has relaxed to a constant. Hence it appears that the main reason for which this formula fails to be valid for large velocities is that, in such a situation, the soliton loses mass.

We have run a set of further simulations, still using anomalous dispersion $D = +1$, varying each parameter in both directions up and down to the boundary of the domain where localized solitons exist (we changed only one parameter at a time, typically multiplying or dividing it by 2, and reached the boundary with an accuracy of about 10%). In every case, both the velocity and the acceleration decrease exponentially, and a quantitative agreement with the analytical approximation is obtained, with the same accuracy as in the special case of Fig. 3. However, the velocity at which the approximation ceases to be valid varies. To discuss this point in a quantitative way, we denote by $V_l$ the velocity at which the accuracy $\Delta F/F$ goes below 0.1. Computation shows that $V_l$ appreciably varies when the parameters are varied and, more precisely, that it depends on the length $\tau$ of the pulse. This length is evaluated using a second-order momentum. Precisely, we set $\tau = 2\sqrt{2}\sigma$, where $\sigma$ is the standard deviation of $\tau$ using the weight $|u|^4$ (rather than $|u|^2$, for which the results depend too strongly on the background noise), which coincides with the half width at $1/e^2$ in the case of a Gaussian pulse. The values of $V_l$ are plotted against $\tau$ in Fig. 5. A linear least-squares fit in logarithmic scales yields $\ln(V_l) = -0.860 - 1.066\tau$, thus very close to $V_l = 0.4/\tau$. This empirical relationship seems to be valid in the whole domain of existence of dissipative solitons.

On the contrary, if we consider normal dispersion $D = -1$, numerical resolution does not allow us to confirm the validity of the approximation. The reason for this is that dissipative solitons with $\beta = 0$ are unstable for normal dispersion [16]. Hence the computation can be performed only for rather large values of $\beta$, for which the braking is very strong and does not quantitatively obey the approximate analysis above, which assumed small values of $\beta$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The velocity $V_l$ at which the accuracy $\Delta F/F$ of the analytic formula (24) goes below 0.1 against the pulse length $\tau$. The crosses give the numerical data, the green dotted line is the result of a linear least-squares fit in logarithmic scales, and the red solid line is the approximation by $V_l = 0.4/\tau$.}
\end{figure}

\section{IV. CONCLUSION}

We have proved that the effect of the finite bandwidth of gain on the soliton of the CGL equation with anomalous dispersion is equivalent to a viscous friction force, provided that this effect is not too large. This opens the way for the construction of a simplified model to describe dissipative soliton interactions as forces between effective particles: Indeed, if we neglect the term which accounts for the finite bandwidth of gain, the CGL equation is translationally invariant, which induces the conservation of the soliton impulsion. Many properties which are essential to build a model of effective particles are ensured by this approximation. When this task is achieved, it will be possible, with the help of the present work, to take the finite bandwidth of gain into account by adding a viscous friction force to the model.

\begin{thebibliography}{9}


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