Motion of solitons in CGL-type equations

Hervé Leblond\textsuperscript{1}, Foued Amrani\textsuperscript{1}, Alioune Niang\textsuperscript{1}, Boris Malomed\textsuperscript{2}, and Valentin Besse\textsuperscript{1}

\textsuperscript{1}Laboratoire de Photonique d’Angers L\varphi A EA 4464, Université d’Angers, France

\textsuperscript{2}Department of Physical Electronics, Tel Aviv University, Israel
The CGL equation is

\[ \frac{\partial E}{\partial z} = \delta E + \left( \beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) E |E|^2 + (\mu + i\nu) E |E|^4 \]

If \( \beta = 0 \), and \( E_0(z,t) \) solution to CGL

\[ E = E_0(z,t - wz) \exp i \left[ Dw t - \left( Dw^2 / 2 \right) z \right] \]

is a solution moving at inverse speed \( w \).
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- $E$: electric field amplitude;
- $\delta$: net linear gain;
- $\beta$: spectral gain bandwidth;
- $D = \pm 1$: dispersion;
- $\varepsilon$: cubic nonlinear gain;
- $\mu$: quintic nonlinear gain;
- $\nu$: 4th-order nonlinear index;
- $z$: number of round-trips;

If $\beta = 0$, and $E_0(z,t)$ solution to CGL

\[ E = E_0(z,t - wz) \exp i \left[ Dwt - \left( \frac{Dw^2}{2} \right) z \right] \]
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If $\beta = 0$, and $E_0(z,t)$ solution to CGL

$$E = E_0(z,t - wz) \exp i \left[ Dwt - \left( D\frac{w^2}{2} \right) z \right]$$

is a solution moving at inverse speed $w$. 

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Motion of solitons in CGL-type equations
\[ \beta \frac{\partial^2 E}{\partial t^2} \] breaks Galilean invariance and prevents any motion of the solitons.

- If \( \beta \neq 0 \): the moving soliton does not exist.

At \( z = 0 \), \( E = E_0 \exp i \Delta \omega t \), \( \beta = 0.55 \) instead of 0.

- Strong breaking due to the limited gain bandwidth.
1. **Motion induced by a continuous wave**
   - Crystal, liquid and gas of solitons
   - Pulse motion due to gain dynamics
   - Injected continuous wave

2. **The finite bandwidth of gain as a viscous friction**
   - Analytical expression of the viscous friction
   - Numerical validation of the approximation

3. **Transverse mobility in the presence of periodic potential**
   - Fundamental soliton
   - Dipoles and vortices
Motion induced by a continuous wave
The finite bandwidth of gain as a viscous friction
Transverse mobility in the presence of periodic potential

- Experiments in mode-locked fiber lasers:
  \[ \implies \text{existence of “soliton gas”} \]
- A large number of solitons in motion.
- Spectral bandwidth of gain is finite: \( \beta \neq 0 \)
- Is the motion it due to the cw component?
- Try to inject cw.

\[
\frac{\partial E}{\partial z} = \delta E + \left( \beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) E |E|^2 \\
+ (\mu + i\nu) E |E|^4 + A \exp(-i\Delta\omega_0 t)
\]

A: amplitude of injected cw; \( \Delta\omega_0 \): frequency shift.
Changing “states of matter”: soliton crystal

-200
-100
0
100
200

t

0
100
200

Z

0
200
400
600
800
1000

Soliton crystal.
\[ \Delta \nu_0 = 1.2. \]
Changing “states of matter”: soliton liquid

Soliton liquid.

$\Delta \nu_0 = 0.9$. 

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Motion of solitons in CGL-type equations
Changing “states of matter”: soliton gas

Soliton gas.
\[ \Delta \nu_0 = 0.8. \]
Changing “states of matter” of solitons

Soliton crystal, $\Delta \nu_0 = 1.2$, 
Soliton liquid, $\Delta \nu_0 = 0.9$, 
Soliton gas, $\Delta \nu_0 = 0.8$. 

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Motion of solitons in CGL-type equations
The different regimes vs detuning $\Delta \omega_0$ and amplitude $A = A_{cw}$ of injected cw.
For a single soliton, input of the form $E = E_0 \exp i\Delta \omega_1 t$, varying $A$, $\Delta \omega_0$ and $\Delta \omega_1 \implies$ Pulse motion.
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The velocity depends on $A$ and $\Delta \omega_0$ (injected cw), but not on $\Delta \omega_1$ (initial speed).

i.e. Speed is entirely determined by injected cw.

Soliton velocity vs $\Delta \nu_0 = \Delta \omega_0 / 2\pi$ for $A = 0.004$ (green dotted), $0.002$ (red dashed), $0.001$ (blue solid line).
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\[ \text{Soliton velocity vs } \Delta \nu_0 = \frac{\Delta \omega_0}{2\pi} \text{ for } A = 0.004 \text{ (green dotted), } 0.002 \text{ (red dashed), } 0.001 \text{ (blue solid line).} \]

\[ \text{Motion is restored but Galilean invariance is not.} \]
“Brownian motion” induced by injected cw

- Soliton velocity is fixed by the cw.
- At higher amplitudes of injected cw:
  More complex nonlinear interaction between cw and solitons.
  \[ \Rightarrow \] The amplitude of cw (radiation) varies with \( t \).
- A tiny variation of the cw component in either amplitude or frequency changes radically the soliton velocity
  \[ \Rightarrow \] apparently random variations of the the soliton speed.
- \[ \Rightarrow \] Erratic motion of solitons, and soliton gas.
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Motion of solitons in CGL-type equations
An integral term accounting for the fast gain dynamics

\[
\frac{\partial E}{\partial z} = \delta E + \left( \beta + i \frac{D}{2} \right) \frac{\partial^2 E}{\partial t^2} + (\varepsilon + i) |E|^2 + (\mu + i\nu) |E|^4
\]

\[
-\Gamma E \int_{-\infty}^{t} (|E|^2 - \langle |E|^2 \rangle) \, dt'
\]

- Represents the decrease of the population inversion (and hence of gain) when stimulated emission occurs.

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Motion of solitons in CGL-type equations
A single pulse (or two-pulse) input is unstable:
New pulses form in front of the input (towards $t < 0$), and quickly disappear.
Unstable pulse emission repeats all along the cavity, \(\rightarrow\) multi-pulse pattern
Solitons move slowly
For larger $\Gamma$, the instability increases: pulses form and vanish faster.

Then, the pulse train does not stabilize any more: pulses are created and vanish permanently.

Generation and vanishing process $\Rightarrow$ effective soliton motion

$\Gamma = 0.03$
- Generation and vanishing process $\Rightarrow$ effective soliton motion
- The inverse velocity $w$ of this motion is very large

$\Gamma = 0.1$. 

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Motion of solitons in CGL-type equations
- Generation and vanishing process $\implies$ effective soliton motion
- The inverse velocity $w$ of this motion is very large

For high values of $\Gamma$, the moving soliton is unstable and vanishes.

Gain dynamics can induce soliton motion.
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Model including external injection, fast gain dynamics, and gain saturation.

\[
\frac{\partial E}{\partial z} = \left(\frac{g_0}{1 + \langle |E|^2 \rangle / I_s} - r\right) E + \left(\beta + i \frac{D}{2}\right) \frac{\partial^2 E}{\partial t^2} \\
+ (\varepsilon + i) E|E|^2 + (\mu + i\nu) E|E|^4 \\
- \Gamma E \int_{-\infty}^{t} (|E|^2 - \langle |E|^2 \rangle) \, dt' + A \exp(-i\Delta\omega_0 t)
\]
Gain saturation limits the number of solitons

⇒ A liquid: condensed phase, which does not fill the box

\[ \Delta \nu_0 = 0.1, \; A = 0.115, \text{ with velocity compensation, } w = -0.05877. \]

- Not a crystal: no phase-locking

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We can still have a soliton gas:

\[ \Delta \nu_0 = 0.1, \ A = 0.120, \ w = -0.07040. \]
Equidistant solitons filling all the box, stable state. When $0.125 \leq A \leq 0.133$.

\[ \Delta \nu_0 = 0.1, \quad A = 0.130, \quad w = -0.02755. \]

Consecutive pulses are phase-locked: a crystal, but the crystal length exactly matches the box length.
In a three bunch pattern, elastic interaction according to the Newton’s cradle scenario

\[ A = 0.130, \Delta \nu_0 = 0.5 \text{ and } w = -0.0024. \]
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Motion of solitons in CGL-type equations
The CGL equation

\[ u_z = \delta u + \left( \beta + i \frac{D}{2} \right) u_{tt} + (\varepsilon + i) u |u|^2 + (\mu + i \nu) u |u|^4. \]

Moving solution: \( u = u_0(t - T, z) e^{i(\omega t - kz)} \)
with \( u_0(t, z) \) solution to CGL with \( \beta = 0 \),
\( T = Vz \), \( \omega = \frac{V}{D} \) and \( k = \frac{V^2}{2D} \).

Perturbative approach: Consider some small non zero \( \beta \).
u a soliton solution,
\[ M = \int_{-\infty}^{+\infty} |u|^2 dt : \text{its mass}, \]
\[ T = \int_{-\infty}^{+\infty} t|u|^2 dt / M : \text{position of its center of mass}. \]
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- The CGL equation
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\[ M = \int_{-\infty}^{+\infty} |u|^2 dt : \text{its mass}, \]
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The velocity of the pulse is then

\[
\frac{dT}{dz} = \frac{1}{M} \int t (u_z u^* + cc) \, dt
\]

\((cc: \text{ complex conjugate}, \, u_z = \partial u/\partial z)\).

Using the CGL equation:

\[
\frac{dT}{dz} = \frac{1}{M} \left( I_1 + \frac{iD}{2} I_2 + \beta I_3 \right) \, dt,
\]

where

\[
I_1 = 2 \int t (\delta |u|^2 + \varepsilon |u|^4 + \mu |u|^6) \, dt,
\]

\[
I_2 = \int t (u_{tt} u^* - cc) \, dt,
\]

\[
I_3 = \int t (u_{tt} u^* + cc) \, dt.
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Using the CGL equation:

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\]

\[
l_2 = \int t \left( u_{tt} u^* - cc \right) dt,
\]

\[
l_3 = \int t \left( u_{tt} u^* + cc \right) dt.
\]
Assumption: $u_0$ is a symmetrical pulse, centered at $t = T$, consequently the function $u_0(t')$, with $t' = t - T$, is even.

Then

$$I_1 = 2 \int t (\delta |u|^2 + \varepsilon |u|^4 + \mu |u|^6) \, dt,$$

becomes

$$I_1 = 2 T \int (\delta |u_0|^2 + \varepsilon |u_0|^4 + \mu |u_0|^6) \, dt'.$$

and so on.

Then we can compute the acceleration $d^2 T / dz^2$:

$$\frac{d^2 T}{dz^2} = \frac{iD}{2M} \frac{dl_2}{dz}.$$
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Using integration by parts and parity we compute a set of integrals.

Finally, we obtain the expression of the force $F = Md^2 T / dz^2$:

$$F = -4\beta \int |u_{0t'}|^2 dt' V$$
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Finally, we obtain the expression of the force $F = Md^2 T / dz^2$:

$$F = -4\beta \int |u_{0t'}|^2 dt' \ V$$
the equation of motion is

\[ M \frac{dV}{dz} = F = -4\beta \int |u_0t'|^2 \, dt' \, V, \]

Hence, the velocity evolves as

\[ V(z) = V(0)e^{-\lambda z} \]

with the decay rate

\[ \lambda = \frac{4\beta}{M} \int |u_0t'|^2 \, dt'. \]
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- An example of calculation

Initial velocity $V_0 = 0.7$ and gain bandwidth coefficient $\beta = 0.004$.

White line: approximate analytical solution
- Good agreement with the numerical solution.
We plot the characteristics of the pulse motion vs $z$

Logarithmic scale.
$V$: velocity; $\gamma$: acceleration and $M$: mass; $F$: force from above theory; $\Delta F/F$: relative difference between $F$ and $M\gamma$.

Parameters: $\omega = 1$, $\beta = 0.0124$. 

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Motion of solitons in CGL-type equations
Effect of finite bandwidth of gain on CGL soliton
with anomalous dispersion
is equivalent to a **viscous friction force**, if it is not too large.

To construct simplified models
to describe CGL soliton interactions
as forces between effective particles:

The lack of Galilean invariance of CGL
was a major difficulty,
since the concept of force is based on it.

With our result, we can approach soliton interaction
in a **conservative frame**.
Then, the finite bandwidth of gain could be treated
as a phenomenological friction force.
Effect of finite bandwidth of gain on CGL soliton with anomalous dispersion is equivalent to a viscous friction force, if it is not too large.

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(2+1)-D spatial Ginzburg-Landau equation:

\[
\frac{\partial u}{\partial Z} = \left[ -\delta + iV(X,Y) + \frac{i}{2} \nabla^2_\perp + (i + \epsilon) |u|^2 - (i\nu + \mu) |u|^4 \right] u,
\]

\( \nabla^2_\perp = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \): the paraxial diffraction

A periodic potential: \( V(X,Y) = -V_0 [\cos(2X) + \cos(2Y)] \)
breaks Galilean invariance

\( \delta = 0.4, \epsilon = 1.85, \mu = 1, \nu = 0.1, V_0 = 1 \),
for which the quiescent fundamental soliton is stable.
Motion induced by a continuous wave
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Fondamental soliton
Dipoles and vortices

The stable fundamental soliton

$|u(X, Y)|$; $|u(X)|$ at $Y = 0$.

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Motion of solitons in CGL-type equations
Input \( u = u_0 \exp (i k_0 \mathbf{R}) \),
with \( \mathbf{R} = (X, Y) \), and \( k_0 = (k_0 \cos \theta, k_0 \sin \theta) \) (0 \( \leq \theta \leq \pi/4 \))

In an amplifier, the factor \((i k_0 \mathbf{R})\)
represents a deviation of the wave vector \( \mathbf{k} \) from the \( Z \)-axis.

Indeed, CGL is derived within the SVEA:
Either \( E = \mathcal{U}(X, Y, Z - \nu T) e^{i(k_x X + k_Y Y + k_Z Z - \omega T)} + c.c., \)
or \( E = u(X, Y, Z - \nu T) e^{i(k_Z Z - \omega T)} + c.c., \)
with \( u(X, Y, Z - \nu T) = \mathcal{U}(X, Y, Z - \nu T) e^{i(k_x X + k_Y Y)}. \)
Equivalent if \( k_x, k_Y \) are small enough.
If $k_0$ is small, the pulse oscillates in the potential site $\{u(X, Z)\}$ in the cross section $Y = 0$, for $k_0 = 1.61$, $\theta = 0$. 
For larger $k_0$, the pulse starts to move.
For larger $k_0$, the pulse starts to move.
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For larger $k_0$, the pulse starts to move...
For larger $k_0$, the pulse starts to move.

$Z = 5.29 \quad Z = 9.39 \quad Z = 17.01 \quad Z = 28.00$

$|u(X,Z)|$ at $Y = 0$, for $k_0 = 1.6878$, $\theta = 0$.

The pulse leaves a copy of it behind it.
Another example, increasing $k_0$:

\[ Z = 5.26 \]
Another example, increasing $k_0$:

$Z = 9.35$
Another example, increasing $k_0$:

$Z = 16.95$
Another example, increasing $k_0$: 

$Z = 24.10$
Another example, increasing $k_0$:

\[ Z = 35.33 \]
Another example, increasing $k_0$:
Another example, increasing $k_0$:

\[ Z = 48.2477, \text{ for } k_0 = 1.694 \]

An arrayed set of 5 fix + 1 moving solitons.
The total number of emitted solitons first grows fast with $k_0$.

- It reaches a maximum of 5 (6 with the initial one)
- Then slowly goes down to 0 (the initial one only)
For the highest $k_0$, the soliton moves freely

$|u(X,Z)|$ at $Y = 0$,

transverse speed, $k_0 = 2.1$, $\theta = 0$.

The soliton velocity increases, approaching a certain limit value.
Periodic elastic collisions

- A moving soliton with one fix soliton

$k_0 = 1.867, \theta = 0.$

- An example of the Newton’s-cradle scenario
1 moving and 5 fix solitons

$|u(X,Z)|$ at $Y = 0$; $k_0 = 1.693$, $\theta = 0$.

An quite complex interaction
- 1 moving and 5 fix solitons

\[ u(X,Z) \] at \( Y = 0; \quad k_0 = 1.693, \theta = 0. \]

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- 1 moving and 5 fix solitons

![Graph showing solitons and their interactions]

\[ |u(X,Z)| \text{ at } Y = 0; \quad k_0 = 1.693, \theta = 0. \]

- An quite complex interaction

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An quite complex interaction
1 moving and 5 fix solitons

\[ |u(X,Z)| \text{ at } Y = 0; \quad k_0 = 1.693, \; \theta = 0. \]

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• An quite complex interaction
1. Motion induced by a continuous wave
   - Crystal, liquid and gas of solitons
   - Pulse motion due to gain dynamics
   - Injected continuous wave

2. The finite bandwidth of gain as a viscous friction
   - Analytical expression of the viscous friction
   - Numerical validation of the approximation

3. Transverse mobility in the presence of periodic potential
   - Fundamental soliton
   - Dipoles and vortices
Motion induced by a continuous wave
The finite bandwidth of gain as a viscous friction
Transverse mobility in the presence of periodic potential

Fondamental soliton
Dipoles and vortices

- Moving the dipole

\[ |u(X, Y)|, \text{ at } Z = 22.410, \text{ for } k_0 = 1.665, \theta = 0. \]

- 5 fix and 1 moving dipoles
Interaction of 1 moving dipole with 1 fix dipole

\[ |u(X,Y,Z)| \text{ at } Y = 0, \text{ for } k_0 = 1.865. \]

- Repeated elastic collisions
- An example of the Newton’s cradle scenario
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- The Newton’s cradle with absorption scenario

\[ |u(X,Y,Z)| \text{ at } Y = 0, \text{ for } k_0 = 1.816. \]

- After several quasi-elastic elastic collisions, the moving dipole is eventually absorbed by the quiescent
- Interaction of 1 moving dipole with 2 fix dipoles

H. Leblond, F. Amrani, A. Niang, B. Malomed, V. Besse
Motion of solitons in CGL-type equations
The Newton’s cradle with absorption scenario

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Interaction of 1 moving dipole with 2 fix dipoles.
Motion induced by a continuous wave
The finite bandwidth of gain as a viscous friction
Transverse mobility in the presence of periodic potential

- Transient Newton's cradle with clearing the obstacle

$|u(X,Y,Z)|$ at $Y = 0$, for $k_0 = 1.884$.

- After several quasi-elastic elastic collisions, the moving dipole absorbs the stationary chain
Transient Newton’s cradle with clearing the obstacle

\[ |u(X,Y,Z)| \text{ at } Y = 0, \text{ for } k_0 = 1.884. \]

After several quasi-elastic elastic collisions, the moving dipole absorbs the stationary chain.
Square-shaped (offsite-centered) vortex.

- It is unstable

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Motion of solitons in CGL-type equations
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Transverse mobility in the presence of periodic potential

- Moving the vortex

A set of fundamental solitons is formed
- For a clockwise rotating vortex, solitons form on the other line

Amplitude at $Z \simeq 300$, for $k_0 = 1.5$, and $\theta = \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$. 
Moving the vortex

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- A set of fundamental solitons is formed
- For a clockwise rotating vortex, solitons form on the other line
- The position of the soliton set depends of the direction of the kick with respect to vortex orientation

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